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CONSTITUTIVE RELATIONS IN CLASSICAL OPTICS IN TERMS OF GEOMETRIC ALGEBRA

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To have a closed system, the Maxwell electromagnetic equations should be supplemented by constitutive relations which describe medium properties and connect primary fields (**E**, **B**) with secondary ones (**D**, **H**). J.W. Gibbs and O. Heaviside introduced the basis vectors {**i**, **j**, **k**} to represent the fields and constitutive relations in the three-dimensional vectorial space. In this paper the constitutive relations are presented in a form of $Cl_{3,0}$ algebra which describes the vector space by three basis vectors $\{\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3\}$ that satisfy Pauli commutation relations. It is shown that the classification of electromagnetic wave propagation phenomena with the help of constitutive relations in this case comes from the structure of $Cl_{3,0}$ itself. Concrete expressions for classical constitutive relations are presented including electromagnetic wave propagation in a moving dielectric.

Keywords: electrodynamics, constitutive relations, light propagation in anisotropic media, geometric algebra, Clifford algebra

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1. Introduction

The Maxwell equations are not complete. The so-called constitutive relations of the medium should be supplemented to determine the electromagnetic field (EM) properties from the Maxwell equations. These relations connect the pair of electric and magnetic fields (**E**, **B**) with the induced pair of fields (**D**, **H**) in material [1, 2, 3, 4]. In general, the constitutive relations have an integral form [5, 6]. To simplify the problem, frequently the constitutive relations are assumed to have a form of linear transformation at a fixed EM wave frequency. Such approximation does not take into account memory and transient processes in the medium.

The constitutive relations may have either a multiplicative or an additive form, for example, the displacement vector may be written as $\mathbf{D} = \varepsilon\mathbf{E}_0$ or as $\mathbf{D} = \mathbf{E}_0 + \mathbf{P}$, where $\mathbf{E}_0$ is the electric field in the vacuum and **P** is the electrical polarization of the material. The second case allows to include permanent polarization. Below only the multiplicative case will be considered.

Two linear forms for multiplicative constitutive relations between the primary and secondary fields are used in the literature. In the so-called Tellegen form [7]

$$\begin{aligned} \mathbf{D} &= \bar{\varepsilon}\mathbf{E} + \bar{\xi}\mathbf{H}, \\ \mathbf{B} &= \bar{\zeta}\mathbf{E} + \bar{\mu}\mathbf{H}, \end{aligned} \tag{1}$$

the primary fields are **E** and **H**. The overbar indicates that, in general, the coupling coeficients are tensors. The Tellegen form is used mainly in electrical engineering [4]. The second form for constitutive relations, also called the Post form [2], is

$$\begin{aligned} \mathbf{D} &= \bar{\varepsilon}\mathbf{E} + \bar{\gamma}\mathbf{B}, \\ \mathbf{H} &= \bar{\beta}\mathbf{E} + \bar{\mu}^{-1}\mathbf{B}. \end{aligned} \tag{2}$$

This form is frequently met in physics. Here the second rank tensors of permittivity $\bar{\varepsilon}$ and permeability $\bar{\mu}$ are collectively called *electromagnetic* coupling coeficients, while $\bar{\beta}$ and $\bar{\mu}^{-1}$ are called *magnetoelectric* coupling coeficients. The sets (1) and (2) can be related by linear transformation except when the determinant becomes zero. For an anisotropic and especially bi-anisotropic medium, however, it is rather difficult to compare and rewrite the intermediate and final formulae if different constitutive relations are used in setting the problem. It should be noted that the relation (2) is more general since it comes from premetric electrodynamics, where no metric tensor of a spacetime is assumed beforehand. More detailed discussion on this subject can be found in the book [3] where physical arguments are given why in the modern physics the pair (**E**, **B**) in the constitutive relation (2) is preferred as a primary field and the pair (**D**, **H**), also called the excitation since it is related with sources, should represent the secondary fields. In this paper the Post form (2) will be used.

We shall consider linear and unbounded media in the frequency domain in terms of geometric algebra (GA). In a tensorial form, similar formulations are summarized in [2] and in the exterior p-form calculus in [3], where it is shown that the constitutive relations are characterized by 36 scalar coeficients in the most general relativistic case. The dyadic analysis of the problem can be found in [4].

The Clifford geometric algebra offers a different approach to the problem. Since the metric of a vector space and allowed involutions of multivectors in GA are predetermined, the main advantage of this algebra comes forth in a simple and clear structuralization of the physical space [8] and, consequently, of the constitutive relations. Furthermore, the coordinate-free representation of physical objects by multivectors, relatively simple manipulations and geometric interpretation of the multivectors allow avoiding superfluous information in intermediate steps that are so characteristic of methods where coordinate representation of physical objects is used, especially in the tensorial calculus, and therefore obscure the physical content of mathematical objects.

At present a number of books at different levels that explain electrodynamics in terms of GA have appeared [9, 10, 11, 12]. In optics and electrodynamics, two of all Clifford algebras are most important, namely, $Cl_{3,0}$ which describes the Euclidean 3D space and $Cl_{1,3}$ which describes the Minkowski 4D space. Respectively, the constitutive relations and optics that follow from these algebras will be called classical (or Galilean) and relativistic (or Minkowskian). The first attempts to construct the constitutive relations in terms of GA multivectors can be found in papers [6, 13, 14, 15]. In [6] the general restrictions in the time domain are considered. In [14] the constitutive relations for isotropic material are formulated in a covariant manner. In [13] the anisotropic medium in the nonorthogonal frame is considered. In [15], the additive relativistic equations are considered. In this paper the constitutive relations in a multiplicative form are presented in the most general form in terms of classical $Cl_{3,0}$ algebra.

In the next Section 2 the Maxwell equations in $Cl_{3,0}$ are summarized. In Section 3 the GA constitutive relations for electromagnetic and magnetoelectric couplings are presented. In Section 4 a compound coupling and its effect on electromagnetic wave propagation are analyzed. A summary of some GA definitions is carried over to Appendix. In the text everywhere the bold fonts will stand for GA vectors and calligraphic fonts for bivectors.

2. Maxwell equations in $Cl_{3,0}$

In $Cl_{3,0}$ algebra, the electric field **E** is represented by the vector (1-grade element) while the magnetic field $\mathcal{B}$ by the bivector (2-grade element), which is an oriented plane:

$$\begin{aligned}\mathbf{E} &= E_1\boldsymbol{\sigma}_1 + E_2\boldsymbol{\sigma}_2 + E_3\boldsymbol{\sigma}_3,\\ \mathcal{B} &= B_1 I\boldsymbol{\sigma}_1 + B_2 I\boldsymbol{\sigma}_2 + B_3 I\boldsymbol{\sigma}_3,\end{aligned} \tag{3}$$

where $\boldsymbol{\sigma}_i$ and $I\boldsymbol{\sigma}_i$, respectively, are elementary vectors and bivectors (orthogonal and oriented planes; see Appendix). E_i and B_i are the scalars. Since the basis vectors $\boldsymbol{\sigma}_i$ are perpendicular to the respective planes $I\boldsymbol{\sigma}_i$, it may be convenient to introduce the magnetic field vector $\mathbf{B} = I\mathcal{B} = -B_1\boldsymbol{\sigma}_1 - B_2\boldsymbol{\sigma}_2 - B_3\boldsymbol{\sigma}_3$, where I is the pseudoscalar of $Cl_{3,0}$ algebra (see Appendix). Thus, **B** is perpendicular to the plane $\mathcal{B}$. We shall note that such equivalence between the vector **B** and the bivector $\mathcal{B}$ exists only in the 3D Euclidean space.

The Maxwell equations in $Cl_{3,0}$ algebra read [9]

$$\nabla \cdot \mathbf{D} = \rho, \tag{4a}$$

$$\nabla \cdot \mathcal{H} + \partial_t\mathbf{D} = -\mathbf{J}, \tag{4b}$$

$$\nabla \wedge \mathbf{E} + \partial_t\mathcal{B} = 0, \tag{4c}$$

$$\nabla \wedge \mathcal{B} = 0. \tag{4d}$$

Here the nabla ∇ operator is defined as a GA vectorial operator

$$\nabla = \boldsymbol{\sigma}_1\frac{\partial}{\partial x} + \boldsymbol{\sigma}_2\frac{\partial}{\partial y} + \boldsymbol{\sigma}_3\frac{\partial}{\partial z}. \tag{5}$$

In the basis $\{\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3\}$ the excitations **D** and $\mathcal{H}$ can be expanded in a similar way as **E** and $\mathcal{B}$ in (3). It should be noted that equations (4a)–(4d) are written without reference to any particular coordinates. In addition, the internal structure of GA which is represented by fundamental involutions (also called the automorphisms) includes space and time reversals automatically [8]. In this reference it is shown explicitly that in the relativistic case represented by $Cl_{1,3}$ algebra the identity, inversion, reversion and Clifford conjugation operations are isomorphic to the group of four which consists of the identity operation, space P and time T reversals, including the combination PT. Thus, in GA the space and time symmetry operations $\{1, P, T, PT\}$ are automatically satisfied.

The classical $Cl_{3,0}$ algebra is isomorphic to even subalgebra $Cl_{1,3}^{(+)}$ of $Cl_{1,3}$, i. e. $Cl_{3,0} \simeq Cl_{1,3}^{(+)}$. The algebra $Cl_{3,0}$ is used in the formulation of classical Newtonian mechanics (see, for example, "New Foundations for Classical Mechanics" by D. Hestenes [16]), electrostatics, magnetostatics and electrodynamics of slowly moving objects. However, the most important advantage of the isomorphism $Cl_{3,0} \simeq Cl_{1,3}^{(+)}$ is that the geometric algebra clearly shows how classical and relativistic descriptions of physics are related. In particular, all of the electromagnetic effects described by the even $Cl_{1,3}^{(+)}$ subalgebra may be attributed to classical electrodynamics, in contrast to those which require full $Cl_{1,3}$ algebra for description. The latter effects may be named as true relativistic effects. In books of physics, especially where vectorial notation is used, there is frequently no clear-cut separation of electrodynamics into classical and relativistic ones. This may lead to misunderstandings and interpretational paradoxes. The present article considers only those constitutive relations that are compatible with the classical physics and respective $Cl_{3,0}$ algebra.

In the absence of sources, when current and electric charge densities are absent, $\mathbf{J} = 0$ and $\rho = 0$, the Maxwell equations for plane and harmonic running fields $\{\mathbf{E}, \mathbf{D}, \mathcal{B}, \mathcal{H}\} = \{\mathbf{E}_0, \mathbf{D}_0, \mathcal{B}_0, \mathcal{H}_0\} \exp[I(\mathbf{k}\cdot\mathbf{r} - \omega t)]$, where I is the pseudoscalar, $\mathbf{k} = \boldsymbol{\sigma}_1 k_x + \boldsymbol{\sigma}_2 k_y + \boldsymbol{\sigma}_y k_z$ is the wave vector, ω is the frequency, and $\{\mathbf{E}_0, \mathbf{D}_0, \mathcal{B}_0, \mathcal{H}_0\}$ are respective amplitudes, can be reduced to the following system of multivector algebraic equations:

$$\mathbf{k}\cdot\mathcal{H} = -\omega\mathbf{D}, \tag{6a}$$

$$\mathbf{k}\wedge\mathbf{E} = \omega\mathcal{B}, \tag{6b}$$

$$\mathbf{k}\cdot\mathbf{D} = 0, \tag{6c}$$

$$\mathbf{k}\cdot\mathcal{B} = 0, \tag{6d}$$

where the subscripts that denote amplitudes were deleted. The wedge and dot symbols denote the outer and inner GA products. The last two equations indicate that the wave vector $\mathbf{k}$ is orthogonal to the displacement vector $\mathbf{D}$ and lies in the $\mathcal{B}$ plane. In agreement with the arguments in [3, 17] the pair $(\mathbf{E}, \mathcal{B})$ is assumed to represent the primary fields. The fields of the second pair $(\mathbf{D}, \mathcal{H})$ will be called the excitations because they are related with the properties of the medium. The relations between the pairs $(\mathbf{E}, \mathcal{B})$ and $(\mathbf{D}, \mathcal{H})$, as mentioned, are called the constitutive relations. Thus, to solve the multivector system (6) the constitutive relations should be assumed. For a linear and homogeneous medium they are the vector-valued functions of the primary fields: $\mathbf{D} = \mathbf{f}_1(\mathbf{E}) + \mathbf{f}_2(\mathcal{B})$ and $\mathbf{H} = -I\mathcal{H} = \mathbf{g}_1(\mathbf{E}) + \mathbf{g}_2(\mathcal{B})$. At first, the anisotropy determined by simple linear relations $\mathbf{D} = \mathbf{f}_1(\mathbf{E})$ and $\mathbf{H} = \mathbf{g}_2(\mathcal{B})$ of electrostatics and magnetostatics will be considered.

Finally, it should be stressed that although the coordinate-free formulation of electrodynamics and constitutive relations in terms of $Cl_{3,0}$ algebra and similar formulations found in the textbooks at first glance may appear just different mathematical descriptions of the same object, however, in principle they are different. The standard vector notation of 3-dimensional (3D) space by basis vectors $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ was introduced more than 100 years ago by J.W. Gibbs and O. Heaviside [18]. The characteristic objects of this space are polar and axial vectors. However, the notion of axial vector (which represents magnetic field) in principle cannot be extended to relativistic 4D or higher dimension spaces. Modern representation of 3D space by GA basis vectors $\{\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3\}$, which are isomorphic to Pauli matrices, mathematically and philosophically has deeper foundation. So the modern geometric algebra conception of 3D classical vector space, in principle, is different from the old Cartesian geometry. The more so, $Cl_{3,0}$ algebra, due to isomorphism, is directly connected with the 4D relativity theory represented by larger relativistic $Cl_{1,3}$ algebra. Thus, in the GA formulation of electrodynamics we have a direct connection between classical and relativity physics.

3. Simple constitutive relations

3.1. Electromagnetic relations

The displacement vector $\mathbf{D}$ induced in an anisotropic dielectric is not parallel to the applied electric field $\mathbf{E}$. If fundamental axes of the dielectric ellipsoid are directed along the basis vectors $\{\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3\}$, then the displacement can be written as

$$\mathbf{D}_\varepsilon = \varepsilon_1 E_1 \boldsymbol{\sigma}_1 + \varepsilon_2 E_2 \boldsymbol{\sigma}_2 + \varepsilon_3 E_3 \boldsymbol{\sigma}_3, \tag{7}$$

which can be cast to form

$$\mathbf{D}_\varepsilon = \varepsilon_1(\mathbf{E}\cdot\boldsymbol{\sigma}_1)\boldsymbol{\sigma}_1 + \varepsilon_2(\mathbf{E}\cdot\boldsymbol{\sigma}_2)\boldsymbol{\sigma}_2 + \varepsilon_3(\mathbf{E}\cdot\boldsymbol{\sigma}_3)\boldsymbol{\sigma}_3, \tag{8}$$

where the inner product acts as a filter which selects the electric field components parallel to fundamental axes. In GA the ellipsoid can be parameterized by two spherical angles θ and φ [19], so the dependence of displacement as a function of angles can be written in a parameterized form as the vector-valued function $\varepsilon : \mathbb{R}^3 \to \mathbb{R}^3$ that maps vectors to vectors:

$$\mathbf{D}_\varepsilon = |\mathbf{E}|\varepsilon(\theta, \varphi) \equiv |\mathbf{E}|\varepsilon(\hat{\mathbf{n}}), \tag{9}$$

where $|\mathbf{E}|$ is the magnitude of the electric field, $\hat{\mathbf{n}} = \mathbf{E}/|\mathbf{E}|$ is the unit vector, $\hat{\mathbf{n}}^2 = 1$, parallel to $\mathbf{E}$. The permittivity ellipsoid is

$$\varepsilon(\hat{\mathbf{n}}) = \sin\theta(\varepsilon_1 \cos\varphi\boldsymbol{\sigma}_1 + \varepsilon_2 \sin\varphi\boldsymbol{\sigma}_2) + \varepsilon_3 \cos\theta\boldsymbol{\sigma}_3. \quad (10)$$

Thus, in GA the permittivity $\varepsilon(\hat{\mathbf{n}})$ is a vector-valued function of unit vector $\hat{\mathbf{n}}$. The value of permittivity in the direction $\hat{\mathbf{n}}||$E when $\mathbf{D}||\hat{\mathbf{m}}$ can be found from

$$\varepsilon_{\hat{\mathbf{m}}\hat{\mathbf{n}}} = \hat{\mathbf{m}} \cdot \varepsilon(\hat{\mathbf{n}}). \quad (11)$$

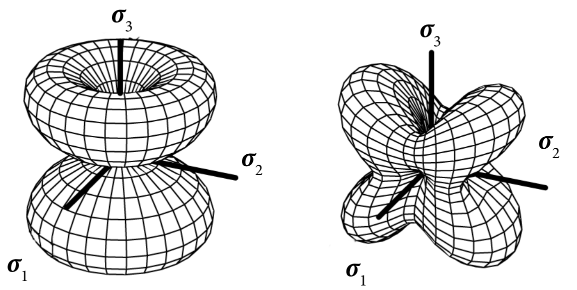

Fig. 1. Shape of the anisotropy factor for uniaxial and biaxial dielectrics with respect to basis vectors $\{\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3\}$ that are parallel to fundamental axes of the dielectric ellipsoid.

From this expression it follows that the components of dielectric matrix are

$$\varepsilon_{ij} = \boldsymbol{\sigma}_i \cdot \varepsilon(\boldsymbol{\sigma}_j). \quad (12)$$

In representation (7) the matrix ε_{ij} is diagonal. If the dielectric ellipsoid has an arbitrary orientation with respect to the basis $\{\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3\}$, Eq. (9) must be generalized. The respective constitutive relation then reads

$$\mathbf{D}_\varepsilon(\mathbf{E}) = \sum_{i,j}^{3} \boldsymbol{\sigma}_i (\mathbf{E}\cdot\boldsymbol{\sigma}_j)\varepsilon_{ij}, \quad \varepsilon_{ij} = \varepsilon_{ij} \cdot \quad (13)$$

The inner product acts as a filter: it selects the jth amplitude $\mathbf{E} \cdot \boldsymbol{\sigma}_j = E_j$ of electric field with respect to the assumed basis $\{\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3\}$ and then associates it with the ith component of the displacement vector. If $\mathbf{E}$ is replaced by the basis vector $\boldsymbol{\sigma}_k$ and Eq. (13) is inner-multiplied by $\boldsymbol{\sigma}_k$, one obtains the permittivity matrix with elements:

$$\varepsilon_{kl} = \boldsymbol{\sigma}_k \cdot \mathbf{D}_\varepsilon(\boldsymbol{\sigma}_l) = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix}, \quad (14)$$

where additional symmetry requirement should be imposed, $\varepsilon_{ij} = \varepsilon_{ji}$. In case when only fundamental axes of the dielectric ellipsoid are known, and one would like to have an arbitrary orientation of the ellipsoid with respect to the basis $\{\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3\}$, then one can address to rotation transformation of the dielectric vector (10) using the GA rotor R [11]:

$$\begin{aligned} \varepsilon'(\hat{\mathbf{n}}) &= R\varepsilon(\hat{\mathbf{n}}) R^{-1}, \\ R &= \exp(-\phi I\boldsymbol{\sigma}_3/2)\exp(-\vartheta I\boldsymbol{\sigma}_2/2), \end{aligned} \quad (15)$$

where ϑ and ϕ are spherical angles of rotation in the planes $I\boldsymbol{\sigma}_2$ and $I\boldsymbol{\sigma}_3$.

The outer product $\mathbf{E} \wedge \mathbf{D}$ is a bivector. For an isotropic dielectric, when $\mathbf{E}||\mathbf{D}$, it is zero. In the general case, if the product $\mathbf{E} \wedge \mathbf{D}$ is multiplied by pseudoscalar $-I$ and normalized, one gets the vector the module of which may be named the anisotropy factor of medium. If normalized, the length of this vector is equal to $\sin\gamma$, where γ is the angle between the vectors $\mathbf{E}$ and $\mathbf{D}$. Figure 1 illustrates the character of the anisotropy factor $\mathbf{E} \wedge \mathbf{D}/|\mathbf{E} \wedge \mathbf{D}|$ for uniaxial and biaxial dielectrics. For the isotropic medium when the permittivity surface is spherical, the anisotropy factor reduces to zero.

3.2. Magnetoelectric relations

There are materials, for example Cr_2O_3, where magnetic field induces electrical displacement [20] and vice versa the electric field induces magnetic displacement. The coupling between electric and magnetic fields is called the magnetoelectric effect. A recent review on the magnetoelectric effect, including history and extensive literature, is given in [20]. Till now the experimentally measured magnetoelectric effect has been observed to be much smaller than that due to the electromagnetic ($\boldsymbol{\varepsilon}$ and $\boldsymbol{\mu}$) coupling [20]. Here we will limit ourselves to the case when dissipation in the material is absent.

In $Cl_{3,0}$ algebra, the electrical response of medium to primary magnetic field can be expressed by a three-component vector $\mathbf{w}_B$. The linear transformation between magnetic field and excitation $\mathbf{D}$ in the medium can then be written in one of coordinate-free forms:

$$\mathbf{D}_\omega = \mathbf{w}_B \cdot \mathcal{B} = \mathbf{w}_B \cdot (I\mathbf{B}) = I(\mathbf{w}_B \wedge \mathbf{B}), \quad (16)$$

where $\mathbf{w}_B = \omega_1\boldsymbol{\sigma}_1 + \omega_2\boldsymbol{\sigma}_2 + \omega_3\boldsymbol{\sigma}_3$ may be called the magnetoelectric coupling vector which, as mentioned, is the material property. In the SI system $\mathbf{w}_B$ is measured in ohms. Figure 2 shows the geometric interpretation of Eq. (16). In the Euclidean space represented by $Cl_{3,0}$ algebra the magnetic field bivector $\mathcal{B}$ can be expressed through the axial magnetic field vector $\mathbf{B} = \mathcal{B}I^{-1} = -\mathcal{B}I$ that is usually introduced in the standard vectorial calculus [18]. In Fig. 2 the vector **B** is drawn as a vertical arrow. Thus, from Eq. (16) it follows that in the Euclidean space the magnetic field **B** induces the electrical displacement $\mathbf{D}_\omega$ that lies in the bivector plane $\mathcal{B}$ and is perpendicular to the vectorial parameter $\mathbf{w}_B$ that characterizes the medium. Finally, it should be noted that within the context of $Cl_{3,0}$ algebra the transformation (16) is the unique one that connects linearly the vector with the bivector.

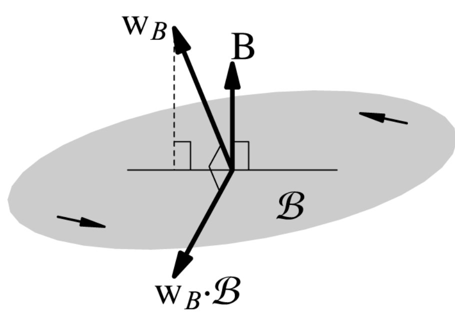

Fig. 2. Graphical representation of the vector-bivector product $\mathbf{D}_\omega = \mathbf{w}_B \cdot \mathcal{B}$ which is a vector that lies in the $\mathcal{B} = I\mathbf{B}$ plane (grey circle) and simultaneously is perpendicular to the material vector $\mathbf{w}_B$ and external magnetic field vector **B**.

The constitutive relation for the permeability ellipsoid can be found from the already obtained results for the permittivity ellipsoid and the duality property between the vectors and bivectors in $Cl_{3,0}$, namely, $\mathcal{H} = I\mathbf{H}$ and $\mathcal{B} = I\mathbf{B}$.

So, in the basis $\{\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3\}$ the magnetic excitation $\mathcal{H}$ can be decomposed:

$$\mathcal{H}_\mu + \mu_1^{-1}B_1I\boldsymbol{\sigma}_1 + \mu_2^{-1}B_2I\boldsymbol{\sigma}_2 + \mu_3^{-1}B_3I\boldsymbol{\sigma}_3. \tag{17}$$

The magnetic terms that are the analogues of electrical ones, Eqs. (9) and (10), now can be easily constructed:

$$\begin{aligned}&\mathcal{H}_\mu = |\mathcal{B}|\boldsymbol{\mu}^{-1}(\theta, \varphi) \equiv |\mathcal{B}|\boldsymbol{\mu}^{-1}(\hat{\mathbf{n}}),\\ &\boldsymbol{\mu}^{-1}(\hat{\mathbf{n}}) = \sin\theta(\mu_1^{-1}\cos\varphi\, I\boldsymbol{\sigma}_1 + \mu_2^{-1}\sin\varphi\, I\boldsymbol{\sigma}_2)\\ &+ \mu_3^{-1}\cos\theta\, I\boldsymbol{\sigma}_3,\end{aligned} \tag{18}$$

where μ_i^{-1} are the lengths of three fundamental axes of the impermeability (inverse permeability) ellipsoid.

In a similar manner the magnetoelectric coupling that is responsible for appearance of magnetic excitation in the presence of external electric field can be obtained from (16) and the duality property. The result is

$$\mathcal{H}_\omega = -\mathbf{w}_E \wedge \mathbf{E}. \tag{19}$$

The dimension of the coupling constant $\mathbf{w}_E$ is $[\mathbf{H}/\mathbf{E}] = [\Omega^{-1}]$. It can be shown that in $Cl_{3,0}$ the structure (19) for magnetoelectric effect is unique as well.

4. Compound constitutive relations and propagation equation

More general relationships between pairs of primary ($\mathbf{D}$, $\mathcal{B}$) and secondary fields ($\mathbf{D}$, $\mathcal{H}$) can be constructed if one sums individuals terms of Eqs. (13), (16), (18) and (19):

$$\begin{aligned}\mathbf{D} &= \mathbf{D}_\varepsilon + \mathbf{D}_\omega,\\ \mathcal{H} &= \mathcal{H}_\mu + \mathcal{H}_\omega,\end{aligned} \tag{20}$$

so that now the both fields, **D** and $\mathcal{B}$, contribute to the excitation of the medium. The compound constitutive relations (20) and Maxwell equations make up a closed system. As an example of application of (20), we shall consider Maxwell equations for an isotropic medium under the following compound constitutive relations of the form

$$\mathbf{D} = \varepsilon\mathbf{E} \pm I\frac{\mathbf{w}}{c\mu} \wedge \mathbf{B}, \tag{21}$$

$$\mathbf{H} = \mp I\frac{\mathbf{w}}{c\mu} \wedge \mathbf{E} + \mathbf{B}/\mu, \tag{22}$$

where $\mathbf{H} = -I\mathcal{H}$ and $c = 1/\sqrt{\mu_0\varepsilon_0}$ is the velocity of light. We shall assume that the electromagnetic coupling is isotropic, i. e. $\varepsilon = \varepsilon_0\varepsilon_r$ and $\mu = \mu_0\mu_r$, where ε_r and μ_r are the relative permittivity and permeability (scalars). The upper and lower signs in (21) and (22) correspond to the dielectric slab movement in opposite directions. It is seen that for propagating waves the magnetoelectric coupling has opposite signs, which means that $c\mu\mathbf{w}_E = (c\mu)^{-1}\mathbf{w}_B \equiv \mathbf{w}$, where **w** is dimensionless. Equations (21) and (22) have the same structure as those for a moving dielectric with velocity much less than the light velocity (Fizeau effect). The equations equivalent to (21) and (22) were obtained earlier from relativistic electrodynamics in [21] in the limit of small velocities of a dielectric slab that carries the electromagnetic wave. Here we see that to get the respective constitutive relations it is enough to address to geometric algebra, isomorphism between $Cl_{3,0}$ and $Cl^{(+)}_{1,3}$ algebras (which has not been used directly here), and to use the linearity of transformation. This allows writing the constitutive relations at once, without any need to resort to the general relativity theory. In addition, the internal structure of GA and the linearity of transformation between the fields and excitations ensure that all possible coupling constants between electric and magnetic fields that are allowed by classical electrodynamics will be taken into account. As we shall see [22], the constitutive relations calculated with relativistic $Cl_{3,0}$ algebra are richer.

The dispersion equation for the plane EM wave will be calculated using the lower signs in (21) and (22). At first, Eq. (6b) is dot-multiplied by $\mathbf{k}$ and rearranged to give $(\mathbf{k} \wedge \mathbf{E}) \cdot \mathbf{k} = \omega \mathbf{B} \cdot \mathbf{k} I$. If $\mathbf{B}$ from Eq. (22) is inserted into this expression, then one finds

$$(\mathbf{k} \wedge \mathbf{E}) \cdot \mathbf{k} = \omega[\mu \mathbf{H} \wedge \mathbf{k} - c^{-1} I(\mathbf{w} \wedge \mathbf{E}) \wedge \mathbf{k}]I. \qquad (23)$$

From Eq. (6a) we have $\mathbf{H} \wedge \mathbf{k} = \omega \mathbf{D} I$. If, in addition, the property $((I\mathbf{w} \wedge \mathbf{E}) \wedge \mathbf{k})I = -(\mathbf{w} \wedge \mathbf{E}) \cdot \mathbf{k}$ is used, we get

$$(\mathbf{k} \wedge \mathbf{E}) \cdot \mathbf{k} = \omega[-\mu\omega \mathbf{D} + c^{-1}(\mathbf{w} \wedge \mathbf{E}) \cdot \mathbf{k}]. \qquad (24)$$

For the isotropic medium $\mathbf{D} = \varepsilon_0 \varepsilon_r \mathbf{E}$ and $\mathbf{k} \cdot (\mathbf{k} \wedge \mathbf{E}) = \mathbf{k}^2 \mathbf{E}$. Then the mixed product can be replaced by $\mathbf{k} \cdot (\mathbf{w} \wedge \mathbf{E}) = (\mathbf{k} \cdot \mathbf{w})\mathbf{E} - (\mathbf{k} \cdot \mathbf{E})\mathbf{w} = (\mathbf{k} \cdot \mathbf{w})\mathbf{E}$ (since $\mathbf{k} \cdot \mathbf{E} = \mathbf{k} \cdot \mathbf{D} \varepsilon_0^{-1} \varepsilon_r^{-1} = 0$), and finally one gets

$$\mathbf{k}^2 \mathbf{E} - \frac{\omega}{c}(\mathbf{k} \cdot \mathbf{w})\mathbf{E} = \frac{\omega^2}{c^2} \mu_r \varepsilon_r \mathbf{E}. \qquad (25)$$

Any electric field $\mathbf{E}$ should satisfy this vectorial equation, therefore from Eq. (25) the scalar dispersion can be obtained:

$$k^2 - \frac{\omega}{\mathrm{c}} k\omega \cos\varphi = \frac{\omega^2}{\mathrm{c}^2} \mu_r \varepsilon_r , \qquad (26)$$

where $k = |\mathbf{k}|$, $w = |\mathbf{w}|$, and φ is the angle between $\mathbf{k}$ and $\mathbf{w}$. Solution of this equation with respect to k,

$$k_{\pm} = \frac{\omega\,(w \cos\varphi \pm \sqrt{w^2 \cos^2\varphi + 4\varepsilon_r \mu_r})}{2c} , \qquad (27)$$

shows that at $w \neq 0$ the lengths of wave vectors $k_{\pm}$ are different for the wave propagating in opposite directions. If the vectors $\mathbf{w}$ and $\mathbf{k}$ are perpendicular, then $k_+ = k_- \equiv k_r$, where $k_{\mathrm{r}} = (\omega/c)\sqrt{\varepsilon_r \mu_r}$. Therefore, the wave propagating in the direction perpendicular to $\mathbf{w}$ is not affected by magnetoelectric coupling at all.

Equation (27) can be rewritten in a form of the sphere shifted along the $\mathbf{w}$ axis (see Fig. 3)

$$k_{\pm} = |\mathbf{k}_{\mathrm{w}}| \cos\varphi \pm \sqrt{\mathbf{k}_{\mathrm{w}}^2 \cos^2\varphi + k_r^2} , \qquad (28)$$

where $k_{\mathrm{r}} = (\omega/c)\sqrt{\varepsilon_{\mathrm{r}} \mu_{\mathrm{r}}}$ is the wave vector in the absence of magnetoelectric coupling and $\mathbf{k}_{\mathrm{w}} = \omega \mathbf{w}/(2c)$. The radius K of sphere depends on $|\mathbf{w}| = w$. In Fig. 3 the sphere is represented by a circle, where K can be found from the condition $K = [k_{\pm}(\varphi = 0) + k_{\pm}(\varphi = \pi)]/2$. The result is

$$K = \frac{\omega\sqrt{w^2 + 4\varepsilon_r \mu_r}}{2c} . \qquad (29)$$

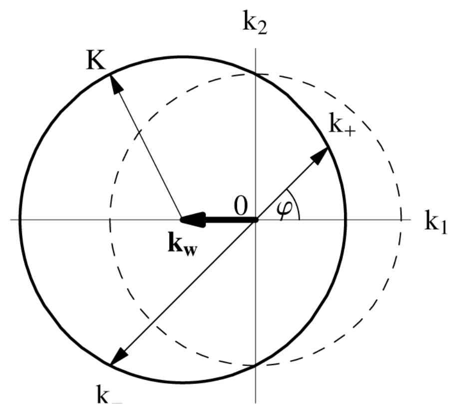

Fig. 3. Constant frequency surfaces in the $\mathbf{k}$ space. The dashed line shows the sphere of radius k_{r} when $\mathbf{w} = 0$. The thick continuous line is a shifted circle of radius K at $\mathbf{w} \neq 0$, Eq. (28). k_+ and k_- are the wave vectors of waves propagating in opposite directions. The constant frequency surfaces have rotational symmetry around the vector $\mathbf{k}_{\mathrm{w}} \| \mathbf{w}$.

Thus we conclude that the wavelength of waves propagating in opposite directions is different. This is also characteristic of chiral media where the waves having clockwise and anticlockwise polarization propagate with different velocities. It should be noted that the magnetoelectric coupling in the general case is the vector and therefore it should be described by three scalar parameters (w_1, w_2, w_3) with respect to permittivity and permeability ellipsoids which, respectively, are also characterized by three scalar parameters.

As it follows from the dispersion relation (27), the phase velocity $v_{\mathrm{ph}} = \omega/k$ and group velocity $v_{\mathrm{gr}} = \partial\omega/\partial k$ are the same:

$$\upsilon_{\mathrm{ph}}=\upsilon_{\mathrm{gr}}\frac{c(-w\cos\varphi\pm\sqrt{w^2\cos^2\varphi+4\varepsilon_{\mathrm r}\mu_{\mathrm r}})}{2\varepsilon_{\mathrm r}\mu_{\mathrm r}}. \quad (30)$$

In metamaterials the velocities υ_{ph} and υ_{gr} have opposite signs. Thus we have found that within classical 3D electrodynamics the moving dielectric cannot transmute to a metamaterial. Expansion of (30) with respect to the coupling w up to linear terms gives $\upsilon_{\mathrm{ph}}=\upsilon_{\mathrm{gr}}\approx(c/n)\,[\pm1-w\cos\varphi/(2n)]$, where plus and minus signs correspond to the opposite directions of EM wave propagation, and $n=\sqrt{\varepsilon_{\mathrm r}\mu_{\mathrm r}}$ is the index of refraction. We see that at a small magnetoelectric coupling the strength of light dragging by a dielectric slab depends on the ratio of strength of the magnetoelectric coupling and the refraction index.

In conclusion, the constitutive relations have been formulated in terms of $Cl_{3,0}$ algebra which connect the pair of primary fields $\mathbf{E}$ and $\mathcal{B}$ with the pair of secondary excitation fields $\mathbf{D}$ and $\mathcal{H}$. In $Cl_{3,0}$ the mixing within individual pairs in either primary or secondary fields is forbidden, since the fields belong to different grades, either to vectors or to bivectors. To include the mixing between different grade fields one must go over to relativistic electrodynamics [22]. In geometric algebra this can be achieved with $Cl_{1,3}$ or $Cl_{3,1}$ algebras, where the primary and secondary fields belong to the same grade, namely, to bivectors. In both algebras there are six basis bivectors which mirror the 6×6 transformation matrix between the components of fields and excitations in relativistic electrodynamics. Thus, GA allows to make a clear distinction between classical or Galilean and relativistic electrodynamics, and to bring in an unambiguous classification between various physical effects. We have also shown that light dragging by a moving dielectric slab can be explained by classical (Galilean) electrodynamics too. However, to explain properties of metamaterials one should address the relativistic electrodynamics, which, as mentioned, can be handled by $Cl_{1,3}$ (or $Cl_{3,1}$) algebra [22].

Appendix: $Cl_{3,0}$ algebra notation

$Cl_{3,0}$ algebra is used for the description of classical physics in the 3D Euclidean space. This space is defined by noncommuting and orthogonal basis vectors $\boldsymbol{\sigma}_i$ which are 1-grade elements that represent oriented lines and anticommute:

$$\begin{cases}\boldsymbol{\sigma}_1^2=\boldsymbol{\sigma}_2^2=\boldsymbol{\sigma}_3^2=1,\\ \boldsymbol{\sigma}_i\boldsymbol{\sigma}_j=-\boldsymbol{\sigma}_j\boldsymbol{\sigma}_i, \qquad \text{if } i\neq j.\end{cases} \quad (31)$$

There are three bivectors (2-grade elements or unit oriented planes),

$$I\boldsymbol{\sigma}_1=\boldsymbol{\sigma}_2\boldsymbol{\sigma}_3,\ I\boldsymbol{\sigma}_3=\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2,\ I\boldsymbol{\sigma}_2=\boldsymbol{\sigma}_3\boldsymbol{\sigma}_1, \quad (32)$$

the squares of which are negative, $(I\boldsymbol{\sigma}_i)^2\equiv I\boldsymbol{\sigma}^2_i=-1$. The pseudoscalar I is a geometric product of all three basis vectors, $I=\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2\boldsymbol{\sigma}_3$, and represents an oriented volume element. It commutes with all basis elements and satisfies $I^2=-1$, $I^{-1}=-I$. The characteristic property of $Cl_{3,0}$ is that the number of elementary vectors and bivectors is the same, therefore, in principle, it is enough to employ one of them. By this reason, in the 3D space the planes can be represented by a respective perpendicular to plane vectors and the standard vector (cross) product of two vectors **a** and **b** can be expressed through the geometric algebra outer product $\mathbf{a}\times\mathbf{b}=-I(\mathbf{a}\wedge\mathbf{b})$. For higher spaces this is not the case because the number of basis planes usually exceeds that of basis vectors. More of GA properties can be found, for example, in books [11, 23].

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SANDAROS RYŠIAI KLASIKINĖJE OPTIKOJE GEOMETRINĖS ALGEBROS POŽIŪRIU

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Santrauka

Kad Maxwello lygčių sistema būtų uždara, ją reikia papildyti sandaros ryšiais, nusakančiais terpės, kurioje sklinda elektromagnetinė banga, savybes ir susiejančiais pirminius elektromagnetinius laukus su antriniais. Straipsnyje pateikti sandaros ryšiai užrašyti $Cl_{3,0}$ algebros, vadinamosios Cliffordo algebra, kalba. Nuo standartinio vektorinio skaičiavimo, plačiai taikomo elektrodinamikoje, ši algebra skiriasi tuo, kad Euklido erdvę sudarantys trys ortai joje tenkina tuos pačius komutacinius sąryšius kaip ir Paulio matricos. Kadangi $Cl_{3,0}$ algebra yra izomorfiška reliatyvistinės $Cl_{1,3}$ algebros lyginiam poalgebriui, manoma, kad $Cl_{3,0}$ algebros matematinis aparatas teisingiau aprašo trimatę Euklido erdvę nei daugiau kaip prieš 100 metų J.W. Gibbso ir O. Heaviside pasiūlyti ortai {**i**, **j**, **k**} ir su jais susietas vektorinis skaičiavimas. Be to, $Cl_{1,3}$ ir $Cl_{3,0}$ algebros aiškiau suskirsto elektrodinamiką į reliatyvistinę ir klasikinę. Straipsnyje nagrinėjami sandaros ryšiai klasikinės elektrodinamikos požiūriu, kai aplinkos atsakas yra tiesinis sužadinimo atžvilgiu ir be vėlinimo. Parodyta, kad tokiu atveju elektromagnetinių bangų sklidimo savybių klasifikacija išeina iš pačios $Cl_{3,0}$ algebros vidinės sandaros ir todėl sandaros ryšiams suformuluoti nėra reikalingi jokie kiti papildomi apribojimai. Pateiktos konkrečios sandaros sąryšių matematinės išraiškos $Cl_{3,0}$ algebros kalba, taip pat jų pagalba išspręstas elektromagnetinės bangos sklidimo judančiame dielektrike uždavinys.