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BOUND STATE INEQUALITY FROM THE SPINLESS SALPETER EQUATION WITH THE YUKAWA POTENTIAL

J.-K. Chen

School of Physics and Information Science, Shanxi Normal University, 041004 Linfen, China
Email: chenjk@sxnu.edu.cn; chenjkphy@yahoo.com

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In this paper, we discuss the bound-state problem for the spinless Salpeter equation with the Yukawa potential. Due to the nonlocal term of the Hamiltonian encountered, we use the eigenfunction for the ground state of the hydrogen atom as a trial function and employ the variational method to solve the spinless Salpeter equation. We derive the upper bounds on the eigenvalues to obtain the bound state inequality. The constraint on the interaction strength α is given, $(2.42m-1.32\mu)\mu/(2.37m^2-m\mu) \le \alpha < 8/(3\pi)$. And the maximum of the screening parameter of the Yukawa potential μ is obtained, $\mu_{\max} = 1.14\, m$.

Keywords: relativistic bound states, Yukawa potential, spinless Salpeter equation

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1. Introduction

It is undoubted that the well-known Yukawa potential [1] is one of the most important potentials in physics and plays an important role in many branches of physics. The Yukawa potential takes the form

$$V(r) = -\alpha \frac{e^{-\mu r}}{r}, \tag{1}$$

where α is the strength of the interaction. In particle physics, the Yukawa potential describes the potential due to scalar particle exchange, and μ is the intermediate particle mass. In plasma physics, the potential (1) is known as the Debye–Hückel potential [2] which describes the ion potential shielded by the presence of neighbouring charged particles for ideal and weakly nonideal plasmas, and $\mu = 1/r_D$ is the reciprocal of the Debye length r_D. In solid state physics, it is known as the Thomas–Fermi potential [3] which represents the effects of a charged particle in a sea of conduction electrons, and μ is the Thomas–Fermi wave vector. In atomic and molecular physics, it describes a screened Coulomb potential [4, 5] due to the cloud of electronic charge around the nucleus, in which μ is the screening parameter and its reciprocal $1/\mu$ describes the effective screening range of the potential.

The Yukawa potential is a short range interaction and it has a distinct difference from the Coulomb potential which is a long-range interaction. The Yukawa potential has a finite number of bound states while the latter has an infinite number. The Schrödinger equation with the Yukawa potential gives the bound state inequality for the existence of the lowest lying S-states [6]

$$\alpha \ge 0.84 \frac{\mu}{m}, \tag{2}$$

where m is the reduced mass. The inequality gives the relation between the interaction strength α, the screening parameter μ and the reduced mass m. In Ref. [7], the very precise value is obtained, $\mu/(m\alpha) \le 1.1906122105(5)$. In Refs. [8–13], the spinless Salpeter equation, the Dirac equation and the Klein–Gordon equation with the Yukawa potential are discussed. In this paper, we present the bound state inequality from the spinless Salpeter equation with the Yukawa potential.

This paper is organized as follows. In Section 2, the spinless Salpeter equation with the Yukawa potential is discussed. By employing the variational method, the bound state inequality is presented. The conclusions are in Section 3.

2. Variational bound on the Yukawa potential

In this section, the Rayleigh–Ritz method is briefly reviewed. Then applying the variational method to the spinless Salpeter equation with the Yukawa potential, the critical value of the interaction strength α is presented.

2.1. Rayleigh–Ritz method

It is known that the min–max principle [14–16] provides the theoretical foundation to derive the rigorous upper bounds on the eigenvalues of some self-adjoint operators which are bounded from below. As a consequence of the min–max principle, the Rayleigh–Ritz method provides a straightforward and efficient means of computing nonincreasing upper bounds on eigenvalues.

Let H be a semibounded self-adjoint operator [14]. Let E_k, $k = 0,1, \ldots$, denote the eigenvalues of H (counting multiplicity) at the bottom of its spectrum with $E_0 \le E_1 \le \ldots$. Let V be a n-dimensional subspace, $V \subset D(H)$, and let P be the orthogonal projection onto V. Let $H_V = PHP$. Let $\hat{E}_0$, $\hat{E}_1,\ldots, \hat{E}_{n-1}$ be the eigenvalues of $HV \restriction V$, ordered by $\hat{E}_0 \le \hat{E}_1 \le \ldots \le \hat{E}_{n-1}$. Then

$$E_i \le \hat{E}_i,\ i = 0,\ldots, n-1. \tag{3}$$

The Rayleigh quotient [17, 18] is defined as

$$F(\psi) = \frac{\langle \psi | H | \psi \rangle}{\langle \psi | \psi \rangle}. \tag{4}$$

There is the Rayleigh principle [15]:

$$E_0 = \min_{\psi \in D(H)} F(\psi). \tag{5}$$

In consequence, there is the inequality given by Rayleigh

$$E_0 \le F(\psi)\ \ (\psi \in D(H)), \tag{6}$$

which is a particular case of the min–max principle, $k = 0$.

2.2. Bound state inequality

The spinless Salpeter equation (SSE) [19–23] is a relativistic extension of the nonrelativistic Schrödinger equation and a well-defined standard approximation to the Bethe–Salpeter equation [24, 25] which is the appropriate tool to describe the bound states within the relativistic quantum field theory.

The spinless Salpeter equation is written in the configuration space as

$$M\psi(\mathbf{r}) = H\psi(\mathbf{r}),\ H = \omega + V(r), \tag{7}$$

where M is the eigenvalue and $\psi(\mathbf{r})$ is the corresponding eigenfunction. $V(r)$ is the Yukawa potential (Eq. (1)). ω is the square-root operator of the relativistic kinetic energy of a particle with mass m and momentum $\mathbf{p}$,

$$\omega = \sqrt{\mathbf{p}^2 + m^2} = \sqrt{-\Delta + m^2}, \tag{8}$$

which is a nonlocal square-root differential operator.

We use the eigenfunction for the ground state of the hydrogen atom as a trial function,

$$\psi(\mathbf{r}) = \sqrt{\frac{\beta^3}{\pi}} \mathrm{e}^{-\beta r},\quad \tilde{\psi}(\mathbf{p}) = \frac{\sqrt{8\beta^5}}{\pi} \frac{1}{(\mathbf{p}^2 + \beta^2)^2}, \tag{9}$$
$$\langle \psi | \psi \rangle = 1,\quad \langle \tilde{\psi} | \tilde{\psi} \rangle = 1,$$

to evaluate the energy expectation value of the Hamiltonian in Eq. (7). Here β is the variational parameter to minimize the expectation value. Using Eqs. (4), (7) and (9), the expectation value of H reads

$$F(\psi) = \langle \psi | H | \psi \rangle = \langle \psi | \omega | \psi \rangle + \langle \psi | V(r) | \psi \rangle. \tag{10}$$

In the above equation, $\langle \omega \rangle$ [8] and $\langle V(r) \rangle$ are

$$\langle\omega\rangle = \frac{2}{3\pi(m^2-\beta^2)^{5/2}} \times \left[\beta\sqrt{m^2-\beta^2}(3m^4-4m^2\beta^2+4\beta^4) + 3m^4(m^2-2\beta^2)\sec^{-1}\frac{m}{\beta}\right], \quad \langle V(r)\rangle = -\frac{4\alpha\beta^3}{(2\beta+\mu)^2}. \tag{11}$$

Eq. (10) can be rewritten as

$$F(\psi) = mM_u, \tag{12}$$

where $\beta' = \beta/m$, $\mu' = \mu/m$,

$$M_u = \frac{2}{3\pi(1-\beta'^2)^{5/2}}\times\left[\beta'\sqrt{1-\beta'^2}(3-4\beta'^2+4\beta'^4) + 3(1-2\beta'^2)\sec^{-1}\frac{1}{\beta'}\right] - \frac{4\alpha\beta'^3}{(2\beta'+\mu')^2}. \tag{13}$$

Using the Rayleigh principle (5) and considering the existence of the bound state, there is the constraints

$$0 \le \min_{\beta'\ge 0} M_u(\beta') \le 1. \tag{14}$$

The mass of the bound state cannot be less than zero which gives a constraint on α. The bound state should exist and the binding energy should be negative for the Yukawa potential, which gives the constraints on α and μ.

In the ultrarelativistic limit, $\beta' \gg 1$, Eq. (13) reduces to

$$M_u = \frac{8}{3\pi}\beta' + \frac{8}{3\pi}\frac{1}{\beta'} - \frac{4\alpha\beta'^3}{(2\beta'+\mu')^2}. \tag{15}$$

In the above equation, the kinematic term takes the large expansion term and the potential term remains unchanged. The Hamiltonian should be bounded from below, there is

$$\alpha \le \frac{8}{3\pi}. \tag{16}$$

This result is independent of μ and m. It is not as tight as the optimum constraint $\alpha < 2/\pi$ [26].

In the nonrelativistic limit, Eq. (13) reduces to

$$M_u = 1 + \frac{\beta'^2}{2} - \frac{4\alpha\beta'^3}{(2\beta'+\mu')^2}. \tag{17}$$

$0 \le \min_{\beta'\ge 0} M_u(\beta')$ gives $\alpha \le \sqrt{2}$. If M_u in Eq. (17) has the lowest point other than the origin of β', i.e.

$$\min_{\beta'\ge 0} M_u(\beta') \le 1, \tag{18}$$

then we can obtain

$$\alpha > \frac{\mu}{m}. \tag{19}$$

Using Eqs. (13) and (14), we obtain the critical value of the interaction strength α. By fitting the calculated data, we obtain the fitting formula for $\alpha_{\min}$:

$$f_1(\mu') = \mu' - 0.012\mu'^2 - 0.189\mu'^3, \tag{20}$$

$$f_2(\mu') = \frac{(2.42-1.32\mu')\mu'}{2.37-\mu'}. \tag{21}$$

The critical value of $\alpha_{\min}$ varies with the screening parameter μ and the mass m, see Fig. 1. We can see that the formula in Eq. (20) is consistent with Eq. (19) as μ' is small. Combining Eqs. (16), (20) and (21), there is the constraint on the interaction strength:

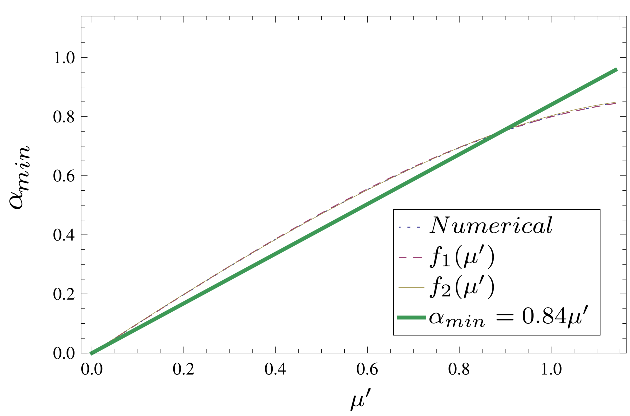

Fig. 1. The critical value of α varies with $\mu' = \mu/m$. The dotted line represents the numerical results obtained from Eq. (13), the dashed line is for the fitting function $f_1(\mu')$, the thin line is for the fitting function $f_2(\mu')$ and the thick line represents the critical value (Eq. (2)) for the Schrödinger equation case.

$$\alpha_{\min} \le \alpha \le \alpha_{\max},$$
$$\alpha_{\min} = f_1(\mu/m), f_2(\mu/m), \quad \alpha_{\max} = \frac{8}{3\pi}. \qquad (22)$$

From Eq. (21), we obtain the condition for the screening parameter:

$$0 \le \mu \le m\left(\frac{11}{12}+\frac{25}{66}\alpha\right) - m\sqrt{\frac{121}{144}-\frac{109}{99}\alpha+\frac{625}{4356}\alpha^2}. \qquad (23)$$

As $\alpha = 8/(3\pi)$, the spinless Salpeter equation predicts as a limit for the S-wave bound state:

$$\left(\frac{\mu}{m}\right)_{\max} = 1.14. \qquad (24)$$

2.3. Discussion

The existence of a bound state is a fundamental and important subject for atomic, molecular and particle physics. The constraints on the interaction strength are summarized in Table 1. For example, for the Dirac equation with the Coulomb potential, the existence of the ground state requires $0 < Z\alpha \le 1$ [10]. From the relation, it is straightforward to make the conclusion that an S-wave bound state composed of one fermion and one point nucleus with the atomic number $Z > 137$ does not exist. In the case of one scalar particle in the Coulomb potential, there is $Z < 69$ [11].

The conditions for the existence of a bound state within the spinless Salpeter equation with the Yukawa potential are Eqs. (22) and (23) which give the constraints on the interaction strength α, the screening parameter μ and the mass of the particle m. The numerical results show that α has a maximum which is independent of μ and m, $\alpha_{\max} = 8/(3\pi)$. As $\alpha > \alpha_{\max}$, the particle in the Yukawa potential will lose its mass due to the binding energy and has negative energy which is in connection with the collapse of the vacuum [11]. As $\alpha < \alpha_{\min}$, the bound state cannot exist due to weak binding. And $\alpha_{\min}$ depends on the ratio between the mass of exchanged boson and the mass of constituent.

The Yukawa potential is a potential induced by a massive intermediate scalar particle with mass μ. If $\mu = 0$, the exchange particle becomes massless and the short-range interaction becomes the long-range Coulomb potential. A larger mass of the exchange scalar particle demands a larger interaction strength to form a bound state. As the interaction strength α is fixed, the exchange particle mass cannot be larger than its critical value, and other particle cannot be bound by this Yukawa potential. According to Eq. (24), $\mu_{\max}$ is related to the mass of constituent m. If the mass of the intermediate scalar particle is greater than $1.14m$, the component with mass m cannot be bound by the Yukawa potential according to the spinless Salpeter equation.

Because of the relativistic kinetic term, the inequality for the spinless Salpeter equation is more complicated than that for the Schrödinger equation which is linear. When $\mu' < 0.8942$, the critical value of α for the spinless Salpeter equation becomes larger than that for the Schrödinger equation. As $\mu' > 0.8942$, the critical value of α for the spinless Salpeter equation becomes smaller than that for the Schrödinger equation.

3. Conclusions

In this paper, the bound state inequality is derived from the spinless Salpeter equation with the Yukawa potential by employing the variational method. Different from the linear inequality for the nonrelativistic Schrödinger equation, the constraint on the interaction strength α for the semirelativistic spinless Salpeter equation is nonlinear. The fitting formula for the inequality

Table 1. The constraints on α for the Schrödinger equation (SE), the Dirac equation (DE), the Klein–Gordon equation (KGE) and the spinless Salpeter equation (SSE) with the Coulomb potential (CP) and the Yukawa potential (YP), respectively.

	SE	DE	KGE	SSE
CP	$0 < \alpha < \sqrt{2}$	$0 < \alpha \le 1$ [10]	$0 < \alpha \le 1/2$ [11]	$0 < \alpha < 1/2$ [27]
YP	$\alpha > 0.84\,\mu/m$ [6]	Figures in Ref. [9]	$\alpha \le 1/2$ [12, 13]	$\alpha_{\min} < \alpha < 8/3\pi$

is $(2.42m-1.32\mu)\mu/(2.37m^2-m\mu) \le \alpha \le 8/(3\pi)$. And the maximum of the screening parameter of the Yukawa potential μ is obtained, $\mu_{max} = 1.14m$.

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SURIŠTOSIOS BŪSENOS NELYGYBĖ REMIANTIS SOLPITERIO LYGTIMI SU JUKAVOS POTENCIALU

J.-K. Chen

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