Lithuanian Journal of Physics, Vol. **62**, No. 1, pp. 1–9 (2022)

QUANTUM JAYNES–CUMMINGS MODEL FOR A TWO-LEVEL SYSTEM WITH EFFECTS OF PARAMETRIC TIME-DEPENDENCES

M. Berrehail [a], N. Benchiheub [a], S. Menouar [b], and J.R. Choi [c]

[a] *Department of Material Sciences, University of Bordj Bou Arreridj, Bordj Bou Arreridj 34000, Algeria*
[b] *Laboratory of Optoelectronics and Compounds (LOC), Department of Physics, Faculty of Science, University of Ferhat Abbas Setif 1, Setif 19000, Algeria*
[c] *Department of Nanoengineering, Kyonggi University, Yeongtong-gu, Suwon, Gyeonggi-do 16227, Republic of Korea*
Email: choiardor@hanmail.net

Received 20 December 2021; accepted 15 February 2022

An approach to exact quantum solutions of the time-dependent two energy level Jaynes–Cummings model with an imaginary photon process is represented in this work. The Lewis–Riesenfeld invariant treatment and the unitary transformation method are used for this purpose. The original Schrödinger equation is reduced to an equivalent solvable one through unitary transformations by using suitable unitary operators. The reduced equation corresponds to a simpler Hamiltonian which is written as a linear combination of the generators of the reduced-dimensional SU(2) algebra. A Hermitian invariant operator is constructed based on the same algebraic formulation and its instantaneous eigenfunctions are obtained. By utilizing such eigenfunctions, the complete quantum wave functions of the system are evaluated. Such wave functions are necessary when we analyze the quantum characteristics of the system.

Keywords: time-dependent two-level atom, Jaynes–Cummings model, unitary transformation, invariant theory, SU(2) algebra

PACS: 03.65.Ge, 03.65.Ca, 03.65.Fd

1. Introduction

The interactions between a two-level atom and a quantized single-mode electromagnetic field have attracted much attention in the physics community up until now, thanks to their potential applicability in the state-of-the-art science [1–6]. The experiments of such interactions reveal many novel quantum effects, such as the quantum collapses and revivals of atomic inversion [7, 8], photon antibunching [9], squeezing of the radiation field [3, 10], inversionless light amplification [11], and inducing a controllable transparency [12, 13]. Two-level atoms are not only used for providing the simplest model for light–matter interactions, but are a kind of potential resources for qubits in quantum information processing, which are indispensable in quantum computation, quantum communication, quantum teleportation, etc. [14–16].

Many quantum problems along this line can be treated by means of the Jaynes–Cummings model [17]. This model provides exact solutions in a wide variety of problems in quantum optics and beyond [6, 18–25]. Furthermore, the researches based on the Jaynes–Cummings model enable us to predict many new features in the limit of strong coupling of the atom and the cavity mode. A notable research subject investigated using this model is the evolution of the geometric phase in quantum systems. For instance, the geometric phase in two energy level atoms which interact with light waves through an imaginary photon process was carried out by adopting the Jaynes–Cummings model [6]. Further, the research for the geometric phase in a generalized Jaynes–Cummings model with double mode operators and phase operators is also found [25]. The Jaynes–Cummings model continuously caught attention from researchers thanks to its relative

simplicity and easiness of its extension to a variety of dynamical systems through analytic expressions and numerical computations. This can be evidenced by the growing abundance of new publications based on this model (see, e.g. Ref. [26] and references therein).

Motivated by the research trend mentioned above, we investigate the quantum characteristics of two-level atoms which undergo time-dependent interaction with radiation fields in a cavity. In order to describe the system, we will use the Jaynes–Cummings model together with the mathematical framework of the invariant methods and the SU(2) algebra. The Jaynes–Cummings model is a very important model, which has been used in many studies such as the quantum spin Hall effect [27], entanglement generation [28] and PT-symmetric system [29]. The detailed calculation in this context will be performed based on the time-dependent unitary transformation.

The quantum invariant theory proposed by Lewis and Riesenfeld [30, 31] will be adopted. This invariant is known as a powerful tool for treating dynamical systems described by time-dependent Hamiltonians. The original basic invariant formulation of the quantum theory has now been generalized so that it can be used more broadly. Actually, the concept of the invariant operator was used to study the exact solutions of various class of time-dependent Schrödinger equations. In the development of our theory, the eigenstates of the time-dependent invariants will be replaced with those of the time-independent invariants via the unitary transformation. Through this procedure, the exact solutions of the time-dependent Schrödinger equation for the system will be derived.

This work is organized as follows. In Section 2, we will formulate the physical problem which describes the Jaynes–Cummings model for two-level atoms interacting with light through an imaginary photon process [6, 32, 33]. In Section 3, we will simplify the Hamiltonian description of our system by introducing the unitary transformation methods. The quantum solutions will be investigated in Section 4 using the invariant methods. The last Section is devoted to the concluding remarks.

2. Mathematical formulation

Let us describe an atom coupled to electromagnetic fields inside a cavity, where the fields are given in the form of a radiation bath. For this purpose, let us denote the position and momentum of *j*th electron in the atom as $\vec{r}_j$ and $\vec{p}_j$, respectively. Then, it is possible to formulate our problem of the system by introducing a generalized time-dependent Hamiltonian of the form

$$H(t)=\sum_j\Bigg[\frac{1}{2m_e}\left(\vec{p}_j-\frac{e}{c}\vec{A}(t)\right)^2 + V_{\text{ext}}(\vec{r}_j,t)+\frac{1}{2}\int(|\vec{E}(t)|^2+|\vec{B}(t)|^2)d^3r + \frac{1}{2}\sum_i U(\vec{r}_j,\vec{r}_i,t)-\frac{e\hbar}{2cm_e}\vec{\sigma}_j\cdot\vec{B}(t)\Bigg], \tag{1}$$

where m_e is the mass of an electron. Here the first term is the minimal coupling, whereas the second term is the collection of the external potentials exerted on each electron. Further, the third, fourth and fifth terms represent the free radiation fields, the interaction energies between electrons, and the spin–field coupling, respectively.

From an appropriate simplification of the Hamiltonian given in Eq. (1) considering the actual situation of the system, we can have the Jaynes–Cummings Hamiltonian. Speaking more precisely, such a model can be derived by considering three approximations. The first approximation is that the distance between electrons is sufficiently large so that the interactions between them can be ignored. The second is that the energies of photons are quite small compared to Rydberg energies, implicating that the spin–field coupling is approximately zero. Finally, as the third approximation, we neglect the term which corresponds to field–field interactions based on the apparent fact that it is relatively weak compared to the electron–field interactions.

From the above approximations, the Hamiltonian operator $H(t)$ describing the system reduces to the sum of three operators: the Hamiltonians, for each, correspond to the free fields $H(t)_{\text{field}}$, the atomic excitation $H(t)_{\text{atom}}$ and the Jaynes–Cummings interaction $H(t)_{\text{interaction}}$. Thus, we have

$$H(t)=H(t)_{\text{field}}+H(t)_{\text{atom}}+H(t)_{\text{interaction}} =\frac{1}{2}\int(|\vec{E}(t)|^2+|\vec{B}(t)|^2)d^3r +\sum_j\left[\frac{p_j^2}{2m_e}+V_{\text{ext}}(\vec{r}_j,t)\right]-\sum_j\frac{e}{m_ec}\vec{A}(t)\cdot\vec{p}_j. \tag{2}$$

The quantified version for this Hamiltonian is given by

$$H(t)=\sum_{k,\upsilon}\hbar\varpi_k(t)a^{\dagger}_{k,\upsilon}(t)a_{k,\upsilon}(t)+\frac{1}{2}\hbar\varpi_0(t)\sigma_z + \sum_{i,j}\sum_{k,\upsilon}\hbar[\Omega(t)b^{\dagger}_j b_i a^{\dagger}_{k,\upsilon}(t)+\Omega^{*}(t)b^{\dagger}_i b_j a_{k,\upsilon}(t)]. \quad (3)$$

$\hbar\varpi_k(t)a^{\dagger}_{k,\upsilon}(t)a_{k,\upsilon}(t)$ in the first term is the energy of the quantized free electromagnetic field associated with the kth mode frequency ϖ_k and the υth element of the polarization among two orthogonal components in the polarization direction, whereas $a_{k,\upsilon}(t)$ and $a^{\dagger}_{k,\upsilon}(t)$ are, respectively, the fermionic annihilation and creation operators in the Heisenberg picture. Notice that quantized fields are composed of a set of harmonic oscillators which have different frequencies. The second term, $\frac{1}{2}\hbar\varpi_0(t)\sigma_z$, is the quantized energy of the matter, where $\varpi_0(t)$ is the atomic transition frequency and σ_z is a Pauli matrix associated with the z axis. Finally, in the interaction term $\Sigma_{i,j}\Sigma_{k,\upsilon}\hbar[\Omega(t)b^{\dagger}_j b_i a^{\dagger}_{k,\upsilon}(t)+\Omega^{*}(t)b^{\dagger}_i b_j a_{k,\upsilon}(t)]$, $\Omega(t)$ is the Rabi frequency in the dipole approximation and b_i is the field operator of the atom. From a substitution of variables via expanding the sum $\Sigma_{i,j}$, it is possible to make replacements $b^{\dagger}_2 b_1 \to \sigma_+$ and $b^{\dagger}_1 b_2 \to \sigma_-$, where the matrices $\sigma_{\pm}$ are two-level atom operators of the form $\sigma_{\pm}=\frac{1}{2}(\sigma_x \pm \mathrm{i}\sigma_y)$. We can easily confirm that the two-level atom operators satisfy the commutation relations, $[\sigma_z, \sigma_{\pm}] = \pm 2\sigma_{\pm}$ and $[\sigma_+, \sigma_-] = \sigma_z$.

In this way, we finally have the Jaynes–Cummings Hamiltonian for the system with the imaginary photon process as

$$H(t)=\hbar\varpi(t)a^{\dagger}a+\frac{1}{2}\hbar\varpi_0(t)\sigma_z + \hbar\Omega_0(t)[a^{\dagger}\sigma_+ \mathrm{e}^{2i\Gamma(t)}+a\sigma_-\mathrm{e}^{-2i\Gamma(t)}], \quad (4)$$

where we have used the relations $\Omega(t)=\Omega_0(t)\mathrm{e}^{2\mathrm{i}\Gamma(t)}$ and $\Omega^{*}(t)=\Omega_0(t)\mathrm{e}^{-2\mathrm{i}\Gamma(t)}$. For convenience, we have dropped the subscripts k and υ in this expression by focusing on the interaction of an atom with a particular mode of the cavity. This Hamiltonian represents the interaction of a two-level atom with a quantized mode of radiation fields in an optical cavity. Such an interaction is the cause of the spontaneous emission and absorption from the atom. If we put $\Omega_0(t)=g(t)$ and $\Gamma(t)=\varpi(t)t$, the Hamiltonian in Eq. (4) reduces to that in Ref. [6].

To understand the quantum behaviour of the system described by the Hamiltonian of Eq. (4), we now solve the corresponding Schrödinger equation, which is of the form

$$\mathrm{i}\hbar\frac{\partial}{\partial t}|\Psi(t)\rangle = H(t)\,|\Psi(t)\rangle. \quad (5)$$

Because the Hamiltonian is a time-dependent form, it is necessary to simplify this equation in order to solve it. We will show how to convert this Hamiltonian to a simple form by means of the unitary transformation in the following section.

3. The unitary transformation

We now carry out a unitary transformation of the original Schrödinger equation in order to reduce it to an equivalent simple one described with a transformed Hamiltonian. Based on a reduced-dimensional SU(2) algebra, the transformed Hamiltonian will be converted into a desired one.

We first consider the following transformation

$$|\Psi(t)\rangle = U(t)|\bar{\Psi}(t)\rangle, \quad (6)$$

where $U(t)$ is a unitary operator that is given by

$$U(t) = \exp[\mathrm{i}\Gamma(t)\sigma_z]. \quad (7)$$

Based on this, the original Hamiltonian given in Eq. (4) can be transformed in a way that

$$\bar{H}(t)=U^{-1}(t)H(t)U(t)-\mathrm{i}U^{-1}(t)\frac{\partial}{\partial t}U(t). \quad (8)$$

Using the relations

$$U^{-1}(t)\sigma_+U(t)=\sigma_+\exp[-\mathrm{i}\Gamma(t)],$$

$$U^{-1}(t)\sigma_-U(t)=\sigma_-\exp[\mathrm{i}\Gamma(t)], \quad (9)$$

the transformed Hamiltonian can be easily evaluated to be

$$\bar{H}(t)=\hbar\varpi(t)a^{\dagger}a+\left[\frac{\hbar\varpi_0(t)}{2}+\dot{\Gamma}(t)\right]\sigma_z + \hbar\Omega_0(t)[a^{\dagger}\sigma_+ + a\sigma_-]. \quad (10)$$

In order to manage this Hamiltonian in an algebraic way, we introduce SU(2) generators of the form

$$\Sigma_1 = \frac{\mathrm{i}}{2\sqrt{\Delta}}(a^\dagger\sigma_+ - a\sigma_-), \tag{11}$$

$$\Sigma_2 = \frac{1}{2\sqrt{\Delta}}(a^\dagger\sigma_+ + a\sigma_-), \tag{12}$$

$$\Sigma_3 = \frac{1}{2}\sigma_z, \tag{13}$$

where

$$\Delta = a^\dagger a + \frac{1}{2}(\mathbf{1} - \sigma_z), \tag{14}$$

while **1** is the identity matrix. These generators satisfy the commutation relations

$$[\Sigma_1, \Sigma_2] = \mathrm{i}\Sigma_3, \tag{15}$$

$$[\Sigma_2, \Sigma_3] = \mathrm{i}\Sigma_1, \tag{16}$$

$$[\Sigma_3, \Sigma_1] = \mathrm{i}\Sigma_2, \tag{17}$$

which are closed. We also consider the relations

$$[\Delta, \Sigma_i] = 0 \qquad (i = 1, 2, 3). \tag{18}$$

Then, the transformed Hamiltonian $\bar{H}(t)$ can be represented as a linear combination of the operators Σ_i, such that

$$\bar{H}(t) = \hbar\varpi(t)\Delta + 2\hbar\Omega_0(t)\sqrt{\Delta}\Sigma_2 + \{\hbar[\varpi_0(t) + \varpi(t)] + 2\dot{\Gamma}(t)\}\Sigma_3 - \frac{\hbar\varpi(t)}{2}. \tag{19}$$

In the context of the Jaynes–Cummings model for a two-level system, the complete set of the exact quantum solutions can be obtained through a generalized quasialgebra in a sub-Hilbert-space. In this regard, the eigenvalue equation for the supersymmetric generator Δ can be written in the form [6, 34]

$$\Delta\begin{pmatrix} |n\rangle \\ |n-2\rangle \end{pmatrix} = n\begin{pmatrix} |n\rangle \\ |n-2\rangle \end{pmatrix}, \tag{20}$$

where n are eigenvalues. Using this, the Hamiltonian, Eq. (19), can further be transformed as

$$\bar{H}(t) = \hbar\varpi(t)n + 2\hbar\Omega_0(t)\sqrt{n}\Sigma_2 + \{\hbar[\varpi_0(t) + \varpi(t)] + 2\dot{\Gamma}(t)\}\Sigma_3 - \frac{\hbar\varpi(t)}{2}. \tag{21}$$

4. Invariant operator and exact quantum solutions

The invariant theory is useful in the research of quantum systems described with a time-dependent Hamiltonian. In many cases along this line, the Hamiltonian is represented by a linear combination of the generators of a certain algebra, like the Heisenberg or Lie algebra, for example. The main advantage of the invariant theory is that it allows one to obtain exact solutions of the Schrödinger equation for a complicated system which involves spatio-temporal differential equations. In this theory, the solutions of the associated Schrödinger equation are represented in terms of the eigenstates of the invariant.

According to the Lewis–Riesenfeld theory [30, 31], a complete set of solutions of the Schrödinger equation for the transformed Hamiltonian are found by using a Hermitian operator $\bar{I}(t)$, which satisfies the Liouville–von Neumann equation:

$$\frac{\mathrm{d}\bar{I}(t)}{\mathrm{d}t} = \frac{\partial\bar{I}(t)}{\partial t} + \frac{1}{\mathrm{i}\hbar}[\bar{I}(t), \bar{H}(t)] = 0. \tag{22}$$

We assume that the set of reference eigenstates $\{|\bar{\lambda}, t\rangle\}$ for the set of operators $\{\bar{I}(t)\}$ are continuous with respect to t. Because all eigenstates are associated with the same eigenvalue, it is possible to write the eigenvalue equation in the form

$$\bar{I}(t)\,|\,\bar{\lambda}, t\rangle = \bar{\lambda}|\bar{\lambda}, t\rangle, \tag{23}$$

where $\bar{\lambda}$ is a time-constant eigenvalue. If the eigenvalue is not degenerate, we can easily obtain the solution of Eq. (23). However, for the case where the eigenvalues are degenerate, Eq. (23) is not always satisfied. In such a case, we must construct the eigenvectors $\{|\bar{\lambda}, t\rangle\}$ which diagonalize the Hermitian operator $(\mathrm{i}\hbar\frac{\partial}{\partial t} - \bar{H})$. This procedure is always possible because the equation of the invariant, Eq. (22), means that $(\mathrm{i}\hbar\frac{\partial}{\partial t} - \bar{H})$ commutes with $\bar{I}(t)$.

The corresponding global phases $\mu_\lambda(t)$ are defined by the relation associated to the wave function $|\bar{\Psi}(t)\rangle$

$$|\,\bar{\Psi}(t)\rangle = \mathrm{e}^{\mathrm{i}\mu_\lambda(t)}\,|\,\bar{\lambda}, t\rangle. \tag{24}$$

While $|\bar{\Psi}(t)\rangle$ satisfies the Schrödinger equation in the transformed system, $\mu_\lambda(t)$ follow the relation

$$\dot{\mu}_{\lambda}(t) = \langle \bar{\lambda}, t | \left(\mathrm{i}\hbar \frac{\partial}{\partial t} - \bar{H} \right) | \bar{\lambda}, t \rangle. \tag{25}$$

A complete set of solutions for the Schrödinger equation is given in terms of the eigenstates of an invariant together with a time-evolving phase. According to the superposition principle, any linear combination of the solutions of the Schrödinger equation is also a solution of the Schrödinger equation.

Let us see a special method for constructing an invariant operator which satisfies the condition in Eq. (22). If an operator which evolves according to the Schrödinger equation is known, it is, in principle, possible to construct appropriate wave states. If an operator $V(t)$ evolves according to the type of the Schrödinger equation

$$\mathrm{i}\hbar \frac{\partial}{\partial t} V(t) = \bar{H}(t) V(t), \tag{26}$$

then, any operator $\bar{I}(t)$, which is defined in a way that

$$\bar{I}(t) = V(t) \tilde{I}(t) V^{-1}(t), \tag{27}$$

where $\tilde{I}(t)$ is *a priori* Hermitian operator, satisfies Eq. (22). To construct an invariant operator according to Eq. (27), we must know $V(t)$ by solving Eq. (26). However, it may be not an easy task to solve Eq. (26). Hence, it is favourable to avoid such a cumbersome task by adopting a more skillful method, i.e. resorting to the Lewis–Riesenfeld method. Nevertheless, one can also exploit Eq. (26) directly in certain particular situations.

Let us now consider a general case where the generators Σ_i follow an l-dimensional closed algebra of Ξ such that

$$\Xi = \{\Sigma_0, \Sigma_1, \Sigma_2, \ldots, \Sigma_l\}, \tag{28}$$

where Σ_0 designates the identity operator. If we assume that the elements in this representation obey the relation

$$[\Sigma_i, \Sigma_j] = \sum_{k=0}^{l} \varepsilon_{ijk} \Sigma_k, \tag{29}$$

the associated Hamiltonian can be written as a linear combination of Σ_i [35]. For the same reason, the invariant operator $\bar{I}(t)$ in our case can be written as a linear combination of the operators given in Eqs. (11–13), such that

$$\bar{I}(t) = \alpha(t)\Sigma_1 + \beta(t)\Sigma_2 + \gamma(t)\Sigma_3, \tag{30}$$

where $\alpha(t)$, $\beta(t)$ and $\gamma(t)$ are time-dependent real coefficients. We note that the operators $\bar{I}(t)$, $\bar{H}(t)$ and $V^{-1}(t)\bar{I}(t)V(t)$ can be generated by using the generators of the same algebra. From the relation $[\Delta, \bar{I}(t)] = 0$, we can confirm that the eigenvectors of the operator Δ are also $|\bar{\lambda}, t\rangle$ as those of $\bar{I}(t)$.

By inserting Eqs. (21) and (30) into Eq. (22), and taking Eq. (29) into account, we see that $\alpha(t)$, $\beta(t)$ and $\gamma(t)$ satisfy the coupled differential equations:

$$\dot{\alpha}(t) = -\{\hbar[\varpi_0(t) + \varpi(t)] + 2\dot{\Gamma}(t)\}\beta(t) + 2\sqrt{n}\hbar\Omega_0(t)\gamma(t), \tag{31}$$

$$\dot{\beta}(t) = \{\hbar[\varpi_0(t) + \varpi(t)] + 2\dot{\Gamma}(t)\}\alpha(t), \tag{32}$$

$$\dot{\gamma}(t) = -2\sqrt{n}\hbar\Omega_0(t)\alpha(t). \tag{33}$$

In terms of the solutions of the above three coupled equations, we can describe the explicit form of $\bar{I}(t)$. If the formulae of time functions are specifically given, the integrations of Eqs. (31), (32) and (33) may be possible. However, in some cases, such integrations would be a little difficult task.

Let us now use well-chosen external parameters for further development. For this, we introduce new operators

$$\Sigma_x = \Sigma_1 + \mathrm{i}\Sigma_2, \tag{34}$$

$$\Sigma_y = \Sigma_1 - \mathrm{i}\Sigma_2, \tag{35}$$

$$\Sigma_z = \Sigma_3. \tag{36}$$

Then, the invariant $\bar{I}(t)$ can be rewritten in terms of them in the form

$$\bar{I}(t) = A(t)\Sigma_x + A^*(t)\Sigma_y + \gamma(t)\Sigma_z, \tag{37}$$

where

$$A(t) = \frac{1}{2}(\alpha(t) - \mathrm{i}\beta(t)) \equiv \kappa \mathrm{e}^{\mathrm{i}\theta}, \tag{38}$$

$$A^*(t) = \frac{1}{2}(\alpha(t) + \mathrm{i}\beta(t)) \equiv \kappa \mathrm{e}^{-\mathrm{i}\theta}, \tag{39}$$

with

$$\kappa = \frac{1}{2}\sqrt{\alpha^2 + \beta^2}, \tag{40}$$

$$\theta(t) = \arctan\frac{\beta(t)}{\alpha(t)}. \tag{41}$$

To solve the eigenvalue equation of the invariant operator $\bar{I}(t)$ which is given by Eq. (37), we regard the unitary transformation

$$|\bar{\lambda}, t\rangle = V(t)|\tilde{\lambda}, t\rangle. \tag{42}$$

Here, the time-dependent unitary operator $V(t)$ is expressed as

$$V(t) = \exp\frac{c(t)}{2}[\Sigma_x e^{-i\theta(t)} - \Sigma_y e^{i\theta(t)}], \tag{43}$$

where the new parameter $c(t)$ is of the form

$$c = \arctan\frac{2\kappa}{\gamma}. \tag{44}$$

The eigenvalue equation for the new invariant operator can be written as

$$\tilde{I}(t)|\tilde{\lambda}, t\rangle = \tilde{\lambda}|\tilde{\lambda}, t\rangle, \tag{45}$$

where $|\tilde{\lambda}, t\rangle$ is the eigenstate. Because eigenvalues do not vary in a unitary transformation, we have $\tilde{\lambda} = \bar{\lambda}$.

From some mathematical manipulations for this transformation, we get

$$\begin{aligned}\Sigma_x &\to V^{-1}(t)\Sigma_x V(t)\\ &= \Sigma_x \cos^2\frac{c}{2} - \Sigma_y e^{2i\theta}\sin^2\frac{c}{2} - \Sigma_z e^{i\theta}\sin c,\end{aligned} \tag{46}$$

$$\begin{aligned}\Sigma_y &\to V^{-1}(t)\Sigma_y V(t)\\ &= \Sigma_y \cos^2\frac{c}{2} - \Sigma_x e^{-2i\theta}\sin^2\frac{c}{2} - \Sigma_z e^{-i\theta}\sin c,\end{aligned} \tag{47}$$

$$\begin{aligned}\Sigma_z &\to V^{-1}(t)\Sigma_z V(t)\\ &= \Sigma_z \cos c + \frac{1}{2}(\Sigma_x e^{-i\theta} + \Sigma_y e^{i\theta}).\end{aligned} \tag{48}$$

Based on this, we finally find the expression of the initial invariant in the form of the operator σ_z:

$$\tilde{I} = V^{-1}(t)\bar{I}(t)V(t) = \sigma_z. \tag{49}$$

In principle, there is no restriction on choosing the formula of the initial invariant so long as our eventual aim is to obtain an invariant operator $\bar{I}(t)$ whose eigenvalue equation is exactly solvable. However, the choice of an initial invariant in a simple form such as Eq. (49) may be very favourable for a succinct description of the system. We note that the initial invariant chosen in Refs. [6] and [34] is also σ_z.

Consequently, the eigenfunctions of the operator $\tilde{I}$ corresponding to the eigenvalues $\tilde{\lambda}_1 = 1$ and $\tilde{\lambda}_2 = -1$ are given by $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$, respectively. As shown in Eq. (18), the operator Δ forms a complete set of observables which commute with σ_z. Thus, by considering the eigenfunctions of the operator Δ, we see that the corresponding normalized eigenfunctions of $\bar{I}(t)$ are given by

$$|\bar{\lambda}, t\rangle = V(t)\begin{pmatrix}|n\rangle\\0\end{pmatrix} \qquad \text{for } \bar{\lambda}_1 = 1, \tag{50}$$

$$|\bar{\lambda}, t\rangle = V(t)\begin{pmatrix}0\\|n-2\rangle\end{pmatrix} \qquad \text{for } \bar{\lambda}_2 = -1. \tag{51}$$

From Eqs. (5), (6), (23), (24) and (25), we obtain two particular solutions of the time-dependent Schrödinger equation of the time-dependent two-level Jaynes–Cummings model as

$$|\Psi(t)\rangle = e^{i\mu_n(t)}U(t)V(t)\begin{pmatrix}|n\rangle\\0\end{pmatrix} \qquad \text{for } \bar{\lambda}_1 = 1, \tag{52}$$

$$|\Psi(t)\rangle = e^{i\mu_n(t)}U(t)V(t)\begin{pmatrix}0\\|n-2\rangle\end{pmatrix} \qquad \text{for } \bar{\lambda}_2 = -1, \tag{53}$$

where the phases are given by

$$\mu_n(t) = \int_0^t W_n(\tau)\mathrm{d}\tau, \tag{54}$$

with

$$\begin{aligned}W_n(\tau) &= 4\sqrt{n}\Omega_0(\tau)\kappa(\tau)\gamma(\tau)\\ &\quad -[\varpi_0(\tau) + \varpi(\tau) + 2\dot{\Gamma}(\tau)]\gamma(\tau)\\ &\quad + n\varpi(\tau) - \dot{\theta}(\tau)c(\tau)\sin c(\tau).\end{aligned} \tag{55}$$

In our work, the parameter $\Gamma(t)$ was taken to be a general form, i.e. it was not written in a special form. If we take $\Gamma(t) = \varpi(t)t$, our development gives the results for the system that was treated in Ref. [6], whereas, by replacing $\Omega_0(t)\exp[2i\Gamma(t)] \to g(t)$, we obtain the solutions of the system given in Ref. [34]. The final solutions, Eqs. (52) and (53), are

useful when we predict the evolution of the probability distribution of the system. Such analytical solutions can be flexibly applied to relevant physical systems through the suitable choice of the time-dependent parameters. The analytical results presented in Eqs. (52)–(55) are also useful for solving similar problems which can use the present model as the first approximation, such as the Tavis–Cummings model [36].

On the one hand, it may be possible to obtain numerical solutions along this line from, for example, the FDTD (finite difference time domain) method. As is well known [37], the FDTD method can be extended to some general situations where the time-dependent Hermitian Hamiltonian involves an explicit imaginary term as well as real ones.

5. Conclusions

Though two-level atoms interacting with radiation fields are potential resources as elements of quantum devices, most of relevant researches so far were devoted to static problems described by a time-independent Hamiltonian. In this paper, the problems related to this have been generalized to a complicated time-dependent Hamiltonian system, by supposing that the parameters such as the electromagnetic field explicitly vary with time.

The complicated mathematical description associated with the original system was reduced to that of a simple system through an alternative treatment. Speaking more concretely, the Hamiltonian given in Eq. (4) was transformed into an equal but a simple one by means of the unitary transformation. To facilitate the derivation of quantum states of the system, we introduced an invariant operator described in terms of SU(2) algebra.

The eigenvalues and eigenstates of the invariant operator were obtained by using the Liouville–von Neumann equation. We showed that the exact wave functions are represented in terms of the eigenstates of the invariant operator and some time-dependent phase factors, $\exp[\mathrm{i}\mu_n(t)]$. As can be seen from Eqs. (52) and (53), they are relatively simple and involve the eigenfunctions of the operator Δ. Such wave functions are basic tools for characterizing the quantum properties of the system.

In conclusion, we have derived the analytical results for the two-level system by transforming the original Hamiltonian into a linear superpositions of generators of the SU(2) group. This belongs to a more general class of operators of the linear combination of unitaries (LCU) proposed in Ref. [38] and the mathematical properties have been extensively studied in Refs. [39, 40]. It has become a powerful tool for designing quantum algorithms [41–43]. It may be an advantage of LCU in solving quantum mechanical problems, both classically as shown in this work, and quantum mechanically as shown in Refs. [41–43].

References

[1] U. Pisipati, I.M. Almakrami, and A. Joshi, Cavity quantum electrodynamics of a two-level atom with modulated fields, Am. J. Phys. **80**(7), 612–620 (2012), https://doi.org/10.1119/1.3703016

[2] R. Salah, A.M. Farouk, A. Farouk, M. Abdel-Aty, H. Eleuch, and A.-S.F. Obada, Entanglement control of two-level atoms in dissipative cavities, Appl. Sci. **10**(4), 1510 (2020), https://doi.org/10.3390/app10041510

[3] T. Abebe, D. Jobir, C. Gashu, and E. Mosisa, Interaction of two-level atom with squeezed vacuum reservoir, Adv. Math. Phys. **2021**(3), 6696253 (2021), https://doi.org/10.1155/2021/6696253

[4] M.S. Abdalla, M.M.A. Ahmed, E.M. Khalil, and A.-S.F. Obada, Dynamics of an adiabatically effective two-level atom interacting with a star-like system, Prog. Theor. Exp. Phys. **2014**(7), 073A02 (2014), https://doi.org/10.1093/ptep/ptu091

[5] D.X. Li and X.Q. Shao, Rapid population transfer of a two-level system by a polychromatic driving field, Sci. Rep. **9**, 9023 (2019), https://doi.org/10.1038/s41598-019-45558-5

[6] G. Yu, Z. Xian, Y.Z. Yong, J.B. Hao, and X.S. Jin, Geometric phase in a two energy level Jaynes-Cummings model with imaginary photon process, Int. J. Theor. Phys. **47**(9), 2279–2284 (2008), https://doi.org/10.1007/s10773-008-9660-y

[7] J.H. Eberly, N.B. Narozhny, and J.J. Sanchez-Mondragon, Periodic spontaneous collapse and revival in a simple quantum model, Phys. Rev. Lett. **44**(20), 1323–1326 (1980), https://doi.org/10.1103/PhysRevLett.44.1323

[8] N.B. Narozhny, J.J. Sanchez-Mondragon, and J.H. Eberly, Coherence versus incoherence: Collapse and revival in a simple quantum model, Phys. Rev. A **23**(1), 236–247 (1981), https://doi.org/10.1103/PhysRevA.23.236

[9] T. Moradi, M.B. Harouni, and M.H. Naderi, Photon antibunching control in a quantum dot and metallic nanoparticle hybrid system with non-Markovian dynamics, Sci. Rep. **8**, 12435 (2018), https://doi.org/10.1038/s41598-018-29799-4

[10] K. Wodkiewicz, P.L. Knight, S.J. Buckle, and S.M. Barnett, Squeezing and superposition states, Phys. Rev. A **35**(6), 2567–2577 (1987), https://doi.org/10.1103/PhysRevA.35.2567

[11] A. Imamoğlu and S.E. Harris, Lasers without inversion: interference of dressed lifetime-broadened states, Opt. Lett. **14**(24), 1344–1346 (1989), https://doi.org/10.1364/OL.14.001344

[12] C. Wang, X. Jiang, G. Zhao, M. Zhang, C.W. Hsu, B. Peng, A.D. Stone, L. Jiang, and L. Yang, Electromagnetically induced transparency at a chiral exceptional point, Nature Phys. **16**, 334–340 (2020), https://doi.org/10.1038/s41567-019-0746-7

[13] B. Peng, Ş.K. Özdemir, W. Chen, F. Nori, and L. Yang, What is and what is not electromagnetically induced transparency in whispering-gallery microcavities, Nature Commun. **5**, 5082 (2014), https://doi.org/10.1038/ncomms6082

[14] E.M. Khalil, K. Berrada, S. Abdel-Khalek, A. Al-Barakaty, and J. Peřina, Entanglement and entropy squeezing in the system of two qubits interacting with a two-mode field in the context of power low potentials, Sci. Rep. **10**, 19600 (2020), https://doi.org/10.1038/s41598-020-76059-5

[15] T. Sowiński, Two-level atom at finite temperature, Acta Phys. Polonica A **116**(6), 994–1005 (2009), https://doi.org/10.12693/APhysPolA.116.994

[16] M.-L. Hu, Teleporting the one-qubit state via two-level atoms with spontaneous emission, J. Phys. B **44**(9), 095502 (2011), https://doi.org/10.1088/0953-4075/44/9/095502

[17] E.T. Jaynes and F.W. Cummings, Comparison of quantum and semiclassical radiation theories with application to the beam maser, Proc. IEEE **51**(1), 89 (1963), https://doi.org/10.1109/PROC.1963.1664

[18] P. Meystre and M.S. Zubairy, Squeezed states in the Jaynes-Cummings model, Phys. Lett. A **89**(8), 390–392 (1982), https://doi.org/10.1016/0375-9601(82)90330-9

[19] R. Short and L. Mandel, Observation of sub-Poissonian photon statistics, Phys. Rev. Lett. **51**(5), 384–387 (1983), https://doi.org/10.1103/PhysRevLett.51.384

[20] J. Eiselt and H. Risken, Quasiprobability distributions for the Jaynes-Cummings model with cavity damping, Phys. Rev. A **43**(1), 346–360 (1991), https://doi.org/10.1103/PhysRevA.43.346

[21] M. Hillery and R.J. Schwartz, Time-averaged properties of the Jaynes-Cummings model: Effects of detuning, Phys. Rev. A **43**(3), 1506–1511 (1991), https://doi.org/10.1103/PhysRevA.43.1506

[22] C.A. Miller, J. Hilsenbeck, and H. Risken, Asymptotic approximations for the *Q* function in the Jaynes-Cummings model, Phys. Rev. A **46**(7), 4323–4334 (1992), https://doi.org/10.1103/physreva.46.4323

[23] K. Matsuo, Origin of splits in *Q* functions for the Jaynes-Cummings model, Phys. Rev. A **50**(1), 649–657 (1994), https://doi.org/10.1103/PhysRevA.50.649

[24] G.S. Agarwal, Master-equation approach to spontaneous emission, Phys. Rev. A **2**(5), 2038–2046 (1970), https://doi.org/10.1103/PhysRevA.2.2038

[25] Z.X. Yu and Z.Y. Jiao, Geometric phase in a generalized Jaynes-Cummings model with double mode operators and phase operators, Int. J. Theor. Phys. **49**(3), 506–511 (2010), https://doi.org/10.1007/s10773-009-0229-1

[26] A.D. Greentree, J. Koch, and J. Larson, Fifty years of Jaynes-Cummings physics, J. Phys. B **46**(22), 220201 (2013), https://doi.org/10.1088/0953-4075/46/22/220201

[27] J. Liu, J.Y. Cao, G. Chen, and Z.Y. Xue, Faithful simulation and detection of quantum spin Hall effect on superconducting circuits, Quantum Eng. **3**(1), e61 (2021), https://doi.org/10.1002/que2.61

[28] S.L. Yang, Y. Zhou, D.Y. Lü, M. Ma, Q.L. Wang, and X.Q. Zhang, Adiabatic preparation of maximum

entanglement in hybrid quantum systems with the Z_2 symmetry, Quantum. Eng. **3**(2), e65 (2021), https://doi.org/10.1002/que2.65

[29] X. Xiao, Q.H. Liao, N.R. Zhou, W.J. Nie, and Y.C. Liu, Tunable optical second-order sideband effects in a parity-time symmetric optomechanical system, Sci. China Phys. Mech. Astron. **63**, 114211 (2020), https://doi.org/10.1007/s11433-020-1559-4

[30] H.R. Lewis, Jr. and W.B. Riesenfeld, Class of exact invariants for classical and quantum time-dependent harmonic oscillators, J. Math. Phys. **9**(11), 1976–1986 (1968), https://doi.org/10.1063/1.1664532

[31] H.R. Lewis, Jr. and W.B. Riesenfeld, An exact quantum theory of the time-dependent harmonic oscillator and of a charged particle in a time-dependent electromagnetic field, J. Math. Phys. **10**(8), 1458–1473 (1969), https://doi.org/10.1063/1.1664991

[32] A.-L. Wang, F.-P. Liu, and Z.-X. Yu, Imaginary photon field effect in the interaction system of multi-atom with single-mode photon field, Int. J. Theor. Phys. **49**(1), 218–223 (2010), https://doi.org/10.1007/s10773-009-0196-6

[33] Y.-X. Qiao, and Z.-X. Yu, Geometric phase in an imaginary photon process, Mod. Phys. **7**(4), 148–154 (2017), https://doi.org/10.12677/MP.2017.74016

[34] J.Q. Shen, H.Y. Zhu, and H. Mao, An approach to exact solutions of the time-dependent supersymmetric two-level three-photon Jaynes-Cummings model, J. Phys. Soc. Jpn. **71**(6), 1440–1444 (2002), https://doi.org/10.1143/JPSJ.71.1440

[35] S.S. Mizrahi, The geometrical phase: An approach through the use of invariants, Phys. Lett A **138**(9), 465–468 (1989), https://doi.org/10.1016/0375-9601(89)90746-9

[36] Y. Jiao, C.H. Bai, D.Y. Wang, S. Zhang, and H.F. Wang, Optical nonreciprocal response and conversion in a Tavis-Cummings coupling optomechanical system, Quantum Eng. **2**(2), e39 (2020), https://doi.org/10.1002/que2.39

[37] I.W. Sudiarta and D.J.W. Geldart, Solving the Schrödinger equation using the finite difference time domain method, J. Phys. A **40**(8), 1885–1896 (2007), https://doi.org/10.1088/1751-8113/40/8/013

[38] G.L. Long, General quantum interference principle and duality computer, Commun. Theor. Phys. **45**(5), 825–844 (2006), https://doi.org/10.1088/0253-6102/45/5/013

[39] S. Gudder, Mathematical theory of duality quantum computers, Quantum Inf. Process. **6**(1), 37–48 (2007), https://doi.org/10.1007/s11128-006-0040-3

[40] Y. Zhang, H.X. Cao, and L. Li, Realization of allowable qeneralized quantum gates, Sci. China Phys. Mech. **5**3(10), 1878–1883 (2010), https://doi.org/10.1007/s11433-010-4078-y

[41] C. Shao, Y. Li, and H. Li, Quantum algorithm design: techniques and applications, J. Syst. Sci. Complex. **32**(1), 375–452 (2019), https://doi.org/10.1007/s11424-019-9008-0

[42] P. Wittek and C. Gogolin, Quantum enhanced inference in Markov logic networks, Sci. Rep. **7(1)**, 45672 (2017), https://doi.org/10.1038/srep45672

[43] S. Wei, H. Li, and G. Long, A full quantum eigensolver for quantum chemistry simulations, Research **2020**, 1486935 (2020), https://doi.org/10.34133/2020/1486935

KVANTINIS JAYNES–CUMMINGS MODELIS DVIEJŲ LYGMENŲ SISTEMAI SU PARAMETRINIŲ LAIKINIŲ PRIKLAUSOMYBIŲ REIŠKINIAIS

M. Berrehail [a], N. Benchiheub [a], S. Menouar [b], J.R. Choi [c]

[a] *Bordž Bu Areridžo universiteto Medžiagų mokslo departamentas, Alž*yras

[b] *Setifo Ferhato Abaso universiteto Mokslo fakultetas, Alž*yras

[c] *Kiongi universiteto Nanoinžinerijos departamentas, Korėjos Respublika*