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NEW GENERALIZED HERMITE POLYNOMIALS WITH THREE VARIABLES OBTAINED VIA QUANTUM OPTICS METHOD AND THEIR APPLICATIONS

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Special polynomials (e.g. Hermite polynomials) are very important for the development of physics and mathematics. As a further extension of ordinary Hermite polynomials, we introduce new generalized Hermite polynomials with three variables and find their generating functions using the operator ordering method in quantum optics. Also, some new operator identities and integral formulas are obtained. As applications, the normalization, Wigner functions and evolutions for certain quantum states are analytically presented. These analytical results can provide conveniences for numerically studying the properties and applications of such quantum states.

Keywords: generalized Hermite polynomial, generating function, operator ordering method, normalization, Wigner function

1. Introduction

As a kind of familiar special polynomials, Hermite polynomials play a key role in physics and mathematics aspects. Generally, single-variable Hermite polynomials $\mathrm{H}_n(x)$ possess the power-series expansions [1]

$$\mathrm{H}_n(x)=\sum_{l=0}^{[n/2]}\frac{n!(-1)^l}{l!(n-2l)!}(2x)^{n-2l}, \tag{1}$$

and the differential representations

$$\mathrm{H}_n(x)=\frac{\partial^n}{\partial t^n}\mathrm{e}^{-t^2+2xt}\,|_{t=0}, \tag{2}$$

where their generating functions read

$$\mathrm{e}^{-t^2+2xt}=\sum_{n=0}^{\infty}\frac{t^2}{n!}\mathrm{H}_n(x). \tag{3}$$

Indeed, Hermite polynomials $\mathrm{H}_n(x)$ are treated as eigenstates for the Hamiltonian in a harmonic oscillator system [2]. Alternatively, two-variable Hermite polynomials $\mathrm{H}_{n,m}(x,y)$ can be expressed as [3, 4]

$$\mathrm{H}_{n,m}(x,y)=\frac{\partial^{n+m}}{\partial t^n\partial t'^m}\mathrm{e}^{-tt'+tx+t'y}\,|_{t=t'=0}=\sum_{l=0}^{\min(n,m)}\frac{n!m!(-1)^l}{l!(n-l)!(m-l)!}x^{n-l}y^{m-l}, \tag{4}$$

$$\mathrm{e}^{-tt'+tx+t'y}=\sum_{n,m=0}^{\infty}\frac{t^n t'^m}{n!m!}\mathrm{H}_{n,m}(x,y), \tag{5}$$

where $\mathrm{H}_{n,m}(x,y)$ can be interpreted as the transition amplitudes of number states for an evolving forced Harmonic oscillator [5] and exist in light propagating plane waves in graded-index media, which helps to show the Talbot effect [6].

In this paper, as a further generalization of the usual two-variable Hermite polynomials $\mathrm{H}_{n,m}(x,y)$, making the substitutions $(-1)^l\to\varsigma^l$ and $x^{n-l}y^{m-l}\to\mathrm{H}_{n-l}(x/2)\mathrm{H}_{m-l}(y/2)$ in the summation of Eq. (4), where ς is an arbitrary real parameter, we have

$$\sum_{l=0}^{\min(n,m)}\frac{\varsigma^l n!m!}{l!(n-l)!(m-l)!}\mathrm{H}_{n-l}\left(\frac{x}{2}\right)\mathrm{H}_{m-l}\left(\frac{y}{2}\right). \tag{6}$$

Thus, similar to Eq. (4), can the summation (6) correspond to new interesting special polynomials? If yes, what is their generating function and what are some useful applications in quantum optics? To tackle these problems, we shall make full use of the operator ordering method in quantum optics to find that the summation (6) is just a new generalized Hermite polynomial with three variables and to present their specific generating functions. Also, we discuss some important applications including the normalization, Wigner function and the evolution of certain quantum states in quantum optics.

2. Deriving new generalized Hermite polynomials and their generating function

To calculate the summation (6), we now make the substitutions $x \to X = \sqrt{2}(a+a^\dagger)$ and $y \to Y = \sqrt{2}(b+b^\dagger)$ in Eq. (4). Since $[X, Y] = 0$, we can introduce the following identity defined as $\mathcal{H}_{n,m}(X, Y; \varsigma)$ (see the Appendix) that is

$$\mathcal{H}_{n,m}(X,Y;\varsigma) = \frac{\partial^{n+m}}{\partial s^n \partial \tau^m} \exp(\varsigma s\tau + sX + \tau Y)\big|_{s=\tau=0} = \sum_{l=0}^{\min(n,m)} \frac{\varsigma^l n!m!}{l!(n-l)!(m-l)!} X^{n-l} Y^{m-l}. \quad (7)$$

Specially, when $\varsigma = -1$, $\mathcal{H}_{n,m}(X, Y; \varsigma)$ becomes the ordinary two-variable Hermite polynomials $\mathrm{H}_{n,m}(X, Y)$. Using the normal ordering products of the operators X^n and Y^m, i.e.

$$X^n = (-\mathrm{i})^n :\mathrm{H}_n\left(\mathrm{i}\frac{X}{2}\right):, \; Y^m = (-\mathrm{i})^m :\mathrm{H}_m\left(\mathrm{i}\frac{Y}{2}\right):, \quad (8)$$

where normal ordering products refer to all the creation operators located on the left of all the annihilation operators, marked with the symbol : [7–11], we thus obtain

$$\mathcal{H}_{n,m}(X,Y;\varsigma) = \sum_{l=0}^{\min(n,m)} \frac{(-\varsigma)^l n!m!(-\mathrm{i})^{n+m}}{l!(n-l)!(m-l)!} \times :\mathrm{H}_{n-l}\left(\mathrm{i}\frac{X}{2}\right)\mathrm{H}_{m-l}\left(\mathrm{i}\frac{Y}{2}\right):. \quad (9)$$

On the other hand, using the Glauber operator formula, that is $\mathrm{e}^{A+B} = \mathrm{e}^A \mathrm{e}^B \mathrm{e}^{-[A,B]/2} = \mathrm{e}^B \mathrm{e}^A \mathrm{e}^{-[B,A]/2}$ subjecting to $[[A, B], A] = [[A, B], B] = 0$, we give the normal ordering product of $\exp(\varsigma s\tau + sX + \tau Y)$ as

$$\exp(\varsigma s\tau + sX + \tau Y) = :\exp(s^2 + \tau^2 + \varsigma s\tau + sX + \tau Y):. \quad (10)$$

So, substituting Eqs. (9) and (10) into Eq. (7), we naturally have

$$\frac{\partial^{n+m}}{\partial s^n \partial \tau^m} :\exp(s^2 + \tau^2 + \varsigma s\tau + sX + X\tau Y):\big|_{s=\tau=0} = \sum_{l=0}^{\min(n,m)} \frac{(-\varsigma)^l n!m!(-\mathrm{i})^{n+m}}{l!(n-l)!(m-l)!} \times :\mathrm{H}_{n-l}\left(\mathrm{i}\frac{X}{2}\right)\mathrm{H}_{m-l}\left(\mathrm{i}\frac{X}{2}\right):. \quad (11)$$

Noting that two sides of Eq. (11) are in normal ordering, by making the substitutions $X \to x$ and $Y \to y$ in Eq. (11), we obtain

$$\frac{\partial^{n+m}}{\partial s^n \partial \tau^m} \exp(s^2 + \tau^2 + \varsigma s\tau + sx + \tau y)\big|_{s=\tau=0} = \sum_{l=0}^{\min(n,m)} \frac{(-\varsigma)^l n!m!(-\mathrm{i})^{n+m}}{l!(n-l)!(m-l)!} \times \mathrm{H}_{n-l}\left(\mathrm{i}\frac{x}{2}\right)\mathrm{H}_{m-l}\left(\mathrm{i}\frac{y}{2}\right). \quad (12)$$

Further, letting $\mathrm{i}x \to x$, $\mathrm{i}y \to y$, $-\mathrm{i}s \to s$ and $-\mathrm{i}\tau \to \tau$, Eq. (12) therefore becomes

$$\frac{\partial^{n+m}}{\partial s^n \partial \tau^m} \exp(-s^2 - \tau^2 + \varsigma s\tau + sx + \tau y)\big|_{s=\tau=0} = \sum_{l=0}^{\min(n,m)} \frac{\varsigma^l n!m!}{l!(n-l)!(m-l)!} \mathrm{H}_{n-l}\left(\frac{x}{2}\right)\mathrm{H}_{m-l}\left(\frac{y}{2}\right). \quad (13)$$

Comparing with Eq. (4), thus we can define new generalized Hermite polynomials $\mathbb{H}_{n,m}(x, y; \varsigma)$ with three variables as

$$\mathbb{H}_{n,m}(x,y;\varsigma) = \frac{\partial^{n+m}}{\partial s^n \partial \tau^m} \exp(-s^2 - \tau^2 + \varsigma s\tau + sx + \tau y)\big|_{s=\tau=0} = \sum_{l=0}^{\min(n,m)} \frac{\varsigma^l n!m!}{l!(n-l)!(m-l)!} \mathrm{H}_{n-l}\left(\frac{x}{2}\right)\mathrm{H}_{m-l}\left(\frac{y}{2}\right), \quad (14)$$

so their generating functions are

$$\exp(-s^2-\tau^2+\varsigma s\tau+sx+\tau y)= \\ =\sum_{n,m=0}^{\infty}\frac{s^n\tau^m}{n!m!}\mathbb{H}_{n,m}(x,y;\varsigma). \quad (15)$$

Obviously, the expansions of the polynomials $\mathbb{H}_{n,m}(x, y; \varsigma)$ and their generating functions are completely different from those of the usual Hermite polynomials $H_n(x)$ and $H_{n,m}(x, y)$. For example, in the case of $\varsigma = 0$, Eq. (14) becomes the product of two single-variable Hermite polynomials, that is $\mathbb{H}_{n,m}(x, y; 0) = H_n(x/2)\, H_m(y/2)$. However, the polynomials $\mathbb{H}_{n,m}(x, y; \varsigma)$ obey some properties (e.g. differential relation and recurrence relation) similar to those of the usual Hermite polynomials $H_{n,m}(x, y)$. Now, the differential relation is taken as an example to illustrate. In terms of the relation $H'_m(x) = 2mH_{m-1}(x)$ belonging to the single-variable Hermite polynomials $H_m(x)$, we can give the identities as

$$\frac{\partial}{\partial x}\mathbb{H}_{n,m}(x,y;\varsigma)=n\mathbb{H}_{n-1,m}(x,y;\varsigma), \\ \frac{\partial}{\partial y}\mathbb{H}_{n,m}(x,y;\varsigma)=m\mathbb{H}_{n,m-1}(x,y;\varsigma), \quad (16)$$

leading to the high-order differential relation of $\mathbb{H}_{n,m}(x, y; \varsigma)$, that is,

$$\frac{\partial^{n'+m'}}{\partial x^{n'}\partial y^{m'}}\mathbb{H}_{n,m}(x,y;\varsigma)= \\ =\frac{n!m!}{(n-n')!(m-m')!}\mathbb{H}_{n-n',m-m'}(x,y;\varsigma). \quad (17)$$

It is thus clear that the polynomials $\mathbb{H}_{n,m}(x, y; \varsigma)$ possess the same forms as the well-known differential relations of the two-variable Hermite polynomials $H_{n,m}(x, y)$ [12]. As verification, using the mathematical method, taking the direct partial differential of arbitrary order on the variables s and τ, we also prove that $\exp(-s^2 - \tau^2 + \varsigma s\tau + sx + \tau y)$ is just the generating function of the generalized Hermite polynomials $\mathbb{H}_{n,m}(x, y; \varsigma)$, as shown in the Appendix.

3. New operator identities and integral formulas related to $\mathbb{H}_{n,m}(x, y; \varsigma)$

In this section, we show how the generalized Hermite polynomials $\mathbb{H}_{n,m}(x, y; \varsigma)$ and their generating functions can be used to deduce some new operator identities and integral formulas. On the one hand, making the substitutions $x \to \mathrm{i}X$ and $y \to \mathrm{i}Y$ in Eq. (14) and adding the term $(-1)^l(-\mathrm{i})^{n+m}$, thus comparing with Eq. (9), we obtain the normal ordering product of $\mathcal{H}_{n,m}(X, Y; \varsigma)$, that is,

$$\mathcal{H}_{n,m}(X,Y;\varsigma)=(-\mathrm{i})^{n+m}:\mathbb{H}_{n,m}(\mathrm{i}X,\mathrm{i}Y;-\varsigma):. \quad (18)$$

From Eqs. (7) and (15), using the Glauber operator formula, we therefore have

$$\sum_{n,m=0}^{\infty}\frac{s^n\tau^m}{n!m!}\mathcal{H}_{n,m}(X,Y;\varsigma)= \\ =\exp(\varsigma s\tau+sX+\tau Y)= \\ =\vdots\exp(-s^2-\tau^2+\varsigma s\tau+sX+\tau Y)\vdots= \\ =\sum_{n,m=0}^{\infty}\frac{s^n\tau^m}{n!m!}\vdots\mathbb{H}_{n,m}(X,Y;\varsigma)\vdots, \quad (19)$$

where the symbol $\vdots\ \vdots$ denotes the anti-normal ordering, having ordering rules that are opposite to the normal ordering. In other words, anti-normal ordering products require that all the annihilation operators are to the left of all the creation operators. Thus, we find that the anti-normal ordering product of $\mathcal{H}_{n,m}(X, Y; \varsigma)$ reads

$$\mathcal{H}_{n,m}(X,Y;\varsigma)=\vdots\,\mathbb{H}_{n,m}(X,Y;\varsigma)\,\vdots. \quad (20)$$

Also, noting the completeness of two-mode coordinate states $|q_1, q_2\rangle$, i.e.

$$1=\iint_{-\infty}^{\infty}\mathrm{d}q_1\mathrm{d}q_2\,|q_1,q_2\rangle\langle q_1,q_2| \\ =\iint_{-\infty}^{\infty}\mathrm{d}q_1\mathrm{d}q_2:\mathrm{e}^{-(q_1-Q_1)^2-(q_2-Q_2)^2}:, \quad (21)$$

where $Q_1=(a+a^\dagger)/\sqrt{2}$ is the coordinate operator of a mode that yields the eigenequation $Q_1|q_1\rangle = q_1|q_1\rangle$, and $Q_2=(b+b^\dagger)/\sqrt{2}$ of b mode satisfies $Q_2|q_2\rangle = q_2|q_2\rangle$ and using Eq. (18), we thus obtain

$$\mathcal{H}_{n,m}(X,Y;\varsigma)= \\ =\iint_{-\infty}^{\infty}\mathrm{d}q_1\mathrm{d}q_2\,|q_1,q_2\rangle\langle q_1,q_2|\mathcal{H}_{n,m}(2q_1 2q_2;\varsigma)= \\ =\frac{1}{\pi}\iint_{-\infty}^{\infty}\mathrm{d}q_1\mathrm{d}q_2:\mathrm{e}^{-(q_1-Q_1)^2-(q_2-Q_2)^2}:\mathcal{H}_{n,m}(2q_1 2q_2;\varsigma)= \\ =(-\mathrm{i})^{n+m}:\mathbb{H}_{n,m}(\mathrm{i}X,\mathrm{i}Y;-\varsigma):. \quad (22)$$

Further, letting $Q_1 \to x$ and $Q_2 \to y$, Eq. (22) can lead to the following new integral formula

$$\frac{1}{\pi}\iint_{-\infty}^{\infty} \mathrm{d}q_1\mathrm{d}q_2 : \mathrm{e}^{-(q_1-x_1)^2-(q_2-y_2)^2} : \mathcal{H}_{n,m}(2q_1 2q_2;\varsigma) = (-\mathrm{i})^{n+m}\mathbb{H}_{n,m}(\mathrm{i}2x,\mathrm{i}2y;-\varsigma). \quad (23)$$

4. Applications of the polynomials $\mathbb{H}_{n,m}(x, y; \varsigma)$

4.1. Normalization

The normalization of a quantum state is of great importance for characterizing the probability of a successful preparation of this state. Using the new generalized Hermite polynomials $\mathbb{H}_{n,m}(x, y; \varsigma)$, we can calculate the normalization of some quantum states, e.g. the photon-modulated state $(ta + ra^\dagger)^k|0\rangle$, where r and t are respectively the ratios of the photon addition and subtraction that yield the relation $r^2 + t^2 = 1$. Indeed, the state $(ta + ra^\dagger)^k|0\rangle$ can be generated theoretically via repeatedly applying a coherent superposition of photon addition and subtraction upon vacuum state for k times [13, 14]. Noting that the normal ordering product of the photon-modulated operator $(ta + ra^\dagger)^k$ reads [14]

$$(ta+ra^\dagger)^k = \left(-\mathrm{i}\sqrt{\frac{rt}{2}}\right)^k : \mathrm{H}_k\left(\mathrm{i}\frac{ta+ra^\dagger}{\sqrt{2rt}}\right):, \quad (24)$$

we can therefore rewrite the state $(ta + ra^\dagger)^k|0\rangle$ as

$$(ta+ra^\dagger)^k|0\rangle = \left(-\mathrm{i}\sqrt{\frac{rt}{2}}\right)^k \mathrm{H}_k\left(\mathrm{i}\frac{ra^\dagger}{\sqrt{2rt}}\right)|0\rangle. \quad (25)$$

So, using the completeness of coherent states $|\beta\rangle$, the generating function in Eq. (3) and the definition of the generalized polynomials $\mathrm{H}_{n,m}(x, y; \varsigma)$, its normalization factor is obtained as

$$\mathcal{D}_k = \frac{|rt|^k}{2^k}\int\frac{\mathrm{d}^2\beta}{\pi}\mathrm{H}_k\left(-\mathrm{i}\frac{r^*\beta}{\sqrt{2r^*t^*}}\right)\mathrm{H}_k\left(\mathrm{i}\frac{r\beta^*}{\sqrt{2rt}}\right)\mathrm{e}^{-|\beta|^2} =$$
$$= \frac{|rt|^k}{2^k}\frac{\partial^{2k}}{\partial s^k\partial\tau^k}\mathrm{e}^{-s^2-\tau^2}\int\frac{\mathrm{d}^2\beta}{\pi}\times$$
$$\times\exp\left(-|\beta|^2 - \mathrm{i}\frac{\sqrt{2r^*}}{\sqrt{t^*}}\tau\beta + \mathrm{i}\frac{\sqrt{2r}}{\sqrt{t}}s\beta^*\right)\Bigg|_{s=\tau=0} =$$
$$= \frac{|rt|^k}{2^k}\frac{\partial^{2k}}{\partial s^k\partial\tau^k}\exp\left(-s^2-\tau^2+\frac{2|r|}{|t|}s\tau\right)\Bigg|_{s=\tau=0} =$$
$$= \frac{|rt|^k}{2^k}\mathrm{H}_{k,k}\left(0,0;\frac{2|r|}{|t|}\right), \quad (26)$$

which is just proportional to the new generalized Hermite polynomials $\mathbb{H}_{k,k}(0, 0; 2|r|/|t|)$ as a function of the ratio of r and t. Obviously, in terms of the expansions of the polynomials $\mathbb{H}_{n,m}(x, y; \varsigma)$ in Eq. (14), the factor $\mathcal{D}_k$ can be directly calculated, which avoids performing the partial differential operations.

4.2. Wigner functions for squeezed number states

The Wigner function for a quantum state in the phase space is an indicator for determining whether this state exhibits non-classicality. Here we calculate the analytical Wigner functions for the squeezed number states via the polynomials $\mathbb{H}_{n,m}(x, y; \varsigma)$ and numerically investigate the non-classicality of the states.

A squeezed number state ρ_s can be generated theoretically via operating the single-mode squeezing operator $\mathcal{S}(r) = \exp[r(a^2-a^{\dagger 2})/2]$ with the squeezing r on a pure number state $\rho_i = |i\rangle\langle i|$. Using the squeezing transformation $\mathcal{S}(r)a^\dagger\mathcal{S}^\dagger(r) = a\cosh r + a^\dagger \sinh r$ and the normal ordering product of the vacuum projector, that is $|0\rangle\langle 0| = :\mathrm{e}^{-a^\dagger a}:$ [15], we easily obtain the normal ordering form of ρ_s, i.e.

$$\rho_s = c : \mathrm{H}_i(c_1a^\dagger)\mathrm{H}_i(c_1a)\exp[c_2(a^{\dagger 2}+a^2) - a^\dagger a] :, \quad (27)$$

where $c = \mathrm{sech}r\,(-\tanh r)^i/(2^i i!)$, $c_1 = 1/\sqrt{-\sinh 2r}$ and $c_2 = (-\tanh r)/2$. In the coherent state representation, the Wigner operator reads $\Delta(\alpha,\alpha^*) = \pi^{-2}\mathrm{e}^{2|\alpha|^2}\int\mathrm{d}^2\alpha'|\alpha'\rangle\langle-\alpha'|\mathrm{e}^{2(\alpha\alpha'^*-\alpha^*\alpha')}$ [8], so the Wigner function for the state ρ_s is obtained as

$$W_s(\alpha) = \mathrm{tr}[\rho_s\Delta(\alpha,\alpha^*)]$$
$$= c\mathrm{e}^{2|\alpha|^2}\int\frac{\mathrm{d}^2\alpha'}{\pi^2}\mathrm{e}^{2(\alpha\alpha'^*-\alpha^*\alpha')}\langle-\alpha'| : \mathrm{H}_i(c_1a^\dagger)\times$$
$$\times\mathrm{H}_i(c_1a)\exp[c_2(a^{\dagger 2}+a^2)-a^\dagger a] : |\alpha'\rangle. \quad (28)$$

Further, using the inner product $\langle-\alpha'|\alpha'\rangle = \mathrm{e}^{-2|\alpha'|^2}$, the generating function (2) and the mathematical integration formula [16]

$$\int \frac{d^2 z}{\pi} \exp\left(\varsigma |z|^2 + \xi z + \eta z^* + g z^2 + h z^{*2}\right) = \\ = \frac{1}{\sqrt{\varsigma^2 - 4gh}} \exp\left(\frac{-\varsigma\xi\eta + \xi^2 h + \eta^2 g}{\xi^2 - 4gh}\right), \qquad (29)$$

which holds for $\mathrm{Re}(\varsigma \pm g \pm h) < 0$, we have

$$W_s(\alpha) = d \frac{\partial^{2i}}{\partial s^i \partial \tau^i} \exp[-d_1^2(s^2 + \tau^2) + \\ + d_2 s\tau + d_3 s + d_3^* \tau]|_{s=\tau=0} = \\ = d d_1^{2i} \mathbb{H}_{i,i}(d_3/d_1, d_3^*/d_1; d_2/d_1^2), \qquad (30)$$

where we have used the standard definition of the polynomials $\mathbb{H}_{n,m}(x, y; \varsigma)$, and the parameters d, d_1, d_2 and d_3 are, respectively,

$$d = \frac{c}{\pi\sqrt{1-4c_2^2}} \exp\left[\left(2 - \frac{4}{1-4c_2^2}\right)|\alpha|^2 + \\ + \frac{4c_2}{1-4c_2^2}(\alpha^2 + \alpha^{*2})\right], \\ d_1 = \left(1 - \frac{4c_1^2 c_2}{1-4c_2^2}\right)^{1/2}, \quad d_2 = -\frac{4c_1^2}{1-4c_2^2}, \\ d_3 = \frac{4c_1(\alpha^* - 2c_2\alpha)}{1-4c_2^2}. \qquad (31)$$

Obviously, the Wigner functions $W_s(\alpha)$ for the squeezed number states are just related to the generalized Hermite polynomials $\mathbb{H}_{i,i}(d_3/d_1, d_3^*/d_1; d_2/d_1^2)$.

Using the generalized Hermite polynomials $\mathbb{H}_{n,m}(x, y; \varsigma)$ as new 'Wolfram' functions like usual Hermite functions, we can more easily and quickly investigate the Wigner functions for the squeezed number states in the phase space. In Fig. 1, we present the Wigner functions for different values of i and r. Obviously, quantum squeezing, as a non-classical property, always occurs and gradually increases with the increase of the parameter r. Also, there is always an upward main peak for an even i, while a downward main peak for an odd i. This means that the Wigner functions for the squeezed number states with an odd i possess stronger negativity, that is, these states have stronger non-classicality.

4.3. Evolutions of squeezed number states for amplitude decay

Amplitude decay as a purely dissipative noise can cause the non-classicality deterioration in the systems. In the interaction picture, the density operator evolution of the system for amplitude decay reads

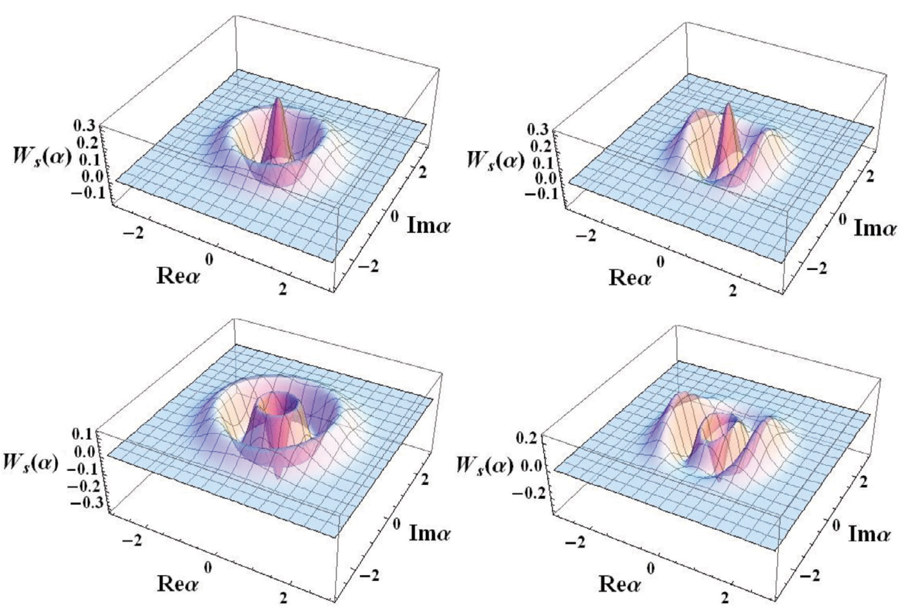

Fig. 1. The Wigner functions for the squeezed number states for different parameters i and r, where the values of (i, r) are, respectively, (a) (2, 0.03), (b) (2, 0.3), (c) (3, 0.03) and (d) (3, 0.5).

$$\frac{\mathrm{d}\rho_t}{\mathrm{d}t} = \kappa(2a\rho_t a^\dagger - a^\dagger a\rho_t - \rho_t a^\dagger a), \quad (32)$$

where κ is the decay rate. Using the thermal entangled state $|\eta\rangle = \mathcal{D}(\eta)|\eta = 0\rangle$ representation, where $|\eta=0\rangle = \mathrm{e}^{a^\dagger\tilde{a}^\dagger}|0,\tilde{0}\rangle$ and $\mathcal{D}(\eta) = \mathrm{e}^{\eta a^\dagger - \eta^* a}$ is the displacement operator, $\tilde{a}^\dagger$ is a fictitious mode accompanying the real photon creation operator $a^\dagger$, the operator master Eq. (10) can be changed into the evolution equation for $\rho|\eta = 0\rangle = |\rho\rangle$, and the Kraus operator-sum representation for ρ_t is analytically obtained as [17, 18]

$$\rho_t = \sum_{j=0}^{\infty} \mathbf{M}_j \rho_0 \mathbf{M}_j^\dagger, \quad (33)$$

where ρ_0 is the density operator for the initial state, and $\mathbf{M}_j = (\mathcal{T}^j/j!)^{1/2}\mathrm{e}^{-\kappa t a^\dagger a}a^j$ is the Kraus operator corresponding to the operator ρ_t for amplitude decay, $\mathcal{T} = 1-\mathrm{e}^{-2\kappa t}$.

Substituting the normal ordering product of the density operator ρ_s in Eq. (27) into Eq. (33) leads to

$$\rho_t = c\sum_{j=0}^{\infty}\frac{\mathcal{T}^j}{j!}\mathrm{e}^{-\kappa t a^\dagger a}a^j :\mathrm{H}_i(c_1 a^\dagger)\mathrm{H}_i(c_1 a)\times$$
$$\times\exp[c_2(a^{\dagger 2} + a^2) - a^\dagger a]: a^{\dagger j}\mathrm{e}^{-\kappa t a^\dagger a}. \quad (34)$$

Inserting the completeness relations of coherent states $|\alpha\rangle$ and $|\beta\rangle$ into Eq. (34) and summing over j, as well as using the generating function (2), we have

$$\rho_t = c\frac{\partial^{2i}}{\partial s^i\partial\tau^i}\mathrm{e}^{-s^2-\tau^2}\iint\frac{\mathrm{d}^2\alpha\mathrm{d}^2\beta}{\pi^2}\exp\big[-|\alpha|^2 - |\beta|^2 +$$
$$+c_2(\alpha^{*2}+\beta^2) + \mathcal{T}\alpha\beta^* + 2c_1\alpha^* s + 2c_1\beta\tau\,]\times$$
$$\times:\exp(\alpha\mathrm{e}^{-\kappa t}a^\dagger + \beta^*\mathrm{e}^{-\kappa t}a - a^\dagger a):|_{s=\tau=0}\,. \quad (35)$$

Further, using the mathematical integration (29) repeatedly and the definition of the polynomials $\mathbb{H}_{n,m}(x, y; \varsigma)$, we obtain

$$\rho_t = \frac{c}{\sqrt{1-4c_2^2\mathcal{T}^2}}\frac{\partial^{2i}}{\partial s^i\partial\tau^i}\exp(-f_1^2 s^2 - f_1^2\tau^2 +$$
$$+f_2 s\tau + f_3 s + f_3^\dagger\tau)|_{s=\tau=0}\times$$
$$\times:\exp\left[\frac{c_2\mathrm{e}^{-2\kappa t}}{1-4c_2^2\mathcal{T}^2}(a^{\dagger 2}+a^2) - \frac{1-4c_2^2 T}{1-4c_2^2\mathcal{T}^2}a^\dagger a\right]:=$$
$$=:f\,f_1^{2i}\mathrm{H}_{i,i}(f_3/f_1, f_3^\dagger/f_1; f_2/f_1^2):, \quad (36)$$

where the parameters f, f_1, f_2 and f_3 read, respectively,

$$f = \frac{\operatorname{sech} r\tanh^i r}{2^i i!\sqrt{1-\mathcal{T}^2\tanh^2 r}}$$
$$\times:\exp\left[\frac{(a^{\dagger 2}+a^2)\mathrm{e}^{-2\kappa t}\tanh r - 2(1-\mathcal{T}\tanh^2 r)a^\dagger a}{2(1-\mathcal{T}^2\tanh^2 r)}\right]:,$$

$$f_1 = \left(\frac{1-\mathcal{T}^2}{1-\mathcal{T}^2\tanh^2 r}\right)^{1/2},\quad f_2 = \frac{4\mathcal{T}}{(1-\mathcal{T}^2\tanh^2 r)\sinh 2r},$$
$$f_3 = \frac{2\mathrm{e}^{-\kappa t}(a^\dagger + \mathcal{T}a\tanh r)}{(1-\mathcal{T}^2\tanh^2 r)\sqrt{\sinh 2r}}. \quad (37)$$

Clearly, the analytical evolutions of the squeezed number states in the amplitude decay process can be simplified as the forms of the generalized Hermite polynomials $\mathbb{H}_{i,i}(f_3/f_1, f_3^\dagger/f_1; f_2/f_1^2)$.

5. Conclusions

In sum, using the operator ordering method in quantum optics, we have introduced new generalized Hermite polynomials with three variables and their generating functions, and presented some new operator identities and integral formulas. As applications, the normalization, Wigner functions and evolutions of some quantum states can be simplified as the forms of the new generalized Hermite polynomials instead of calculating the high-order partial differential, which brings us conveniences for further discussing their properties and applications. In the near future, we believe that the new generalized Hermite polynomials could be used more widely in the fields of physics and mathematics like the usual Hermite polynomials, and more new special polynomials with multiple variables can also be found.

Appendix

The derivation of Eq. (7) reads as follows:

$$\mathcal{H}_{n,m}(x, Y; \varsigma) =$$
$$= \frac{\partial^{n+m}}{\partial s^n\partial\tau^m}\exp(\varsigma s\tau + sX + \tau Y)|_{s=\tau=0} =$$
$$= \frac{\partial^n}{\partial s^n}\mathrm{e}^{sX}\frac{\partial^m}{\partial\tau^m}\exp[(\varsigma s + Y)\tau]|_{s=\tau=0} =$$
$$= \frac{\partial^n}{\partial s^n}[\mathrm{e}^{sX}(\varsigma s + Y)^m]|_{s=0} =$$

$$=\sum_{l=0}\binom{n}{l}\frac{\partial^l}{\partial s^l}(\varsigma s+Y)^m\frac{\partial^{n-l}}{\partial s^{n-l}}\mathrm{e}^{sX}\bigg|_{s=0}=$$

$$=\sum_{l=0}^{\min(n,m)}\frac{\varsigma^l n!m!}{l!(n-l)!(m-l)!}X^{n-l}Y^{m-l}.$$

The derivation of Eq. (14) is the following:

$$\frac{\partial^{n+m}}{\partial s^n\partial\tau^m}\exp(-s^2-\tau^2+\varsigma s\tau+sx+\tau y)|_{s=\tau=0}=$$

$$=\frac{\partial^n}{\partial s^n}\exp(-s^2+sx)\frac{\partial^m}{\partial\tau^m}\exp[-\tau^2+(y+\varsigma s)\tau]|_{s=\tau=0}=$$

$$=\frac{\partial^n}{\partial s^n}\left[\exp(-s^2+sx)\,\mathrm{H}_m\left(\frac{y+\varsigma s}{2}\right)\right]\bigg|_{s=0}=$$

$$=\sum_{l=0}^{\min(n,m)}\binom{n}{l}\frac{\partial^{n-l}}{\partial s^{n-l}}\exp(-s^2+sx)\frac{\partial^n}{\partial s^n}\mathrm{H}_m\left(\frac{y+\varsigma s}{2}\right)\bigg|_{s=0}=$$

$$=\sum_{l=0}^{\min(n,m)}\binom{n}{l}\binom{m}{l}l!\varsigma^l\mathrm{H}_{n-l}\left(\frac{x}{2}\right)\mathrm{H}_{m-l}\left(\frac{y}{2}\right)\equiv\mathbb{H}_{n,m}(x,y;\varsigma).$$

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NAUJI APIBENDRINTIEJI ERMITO POLINOMAI SU TRIMIS KINTAMAISIAIS, GAUTI KVANTINĖS OPTIKOS METODU, IR JŲ TAIKYMAI

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